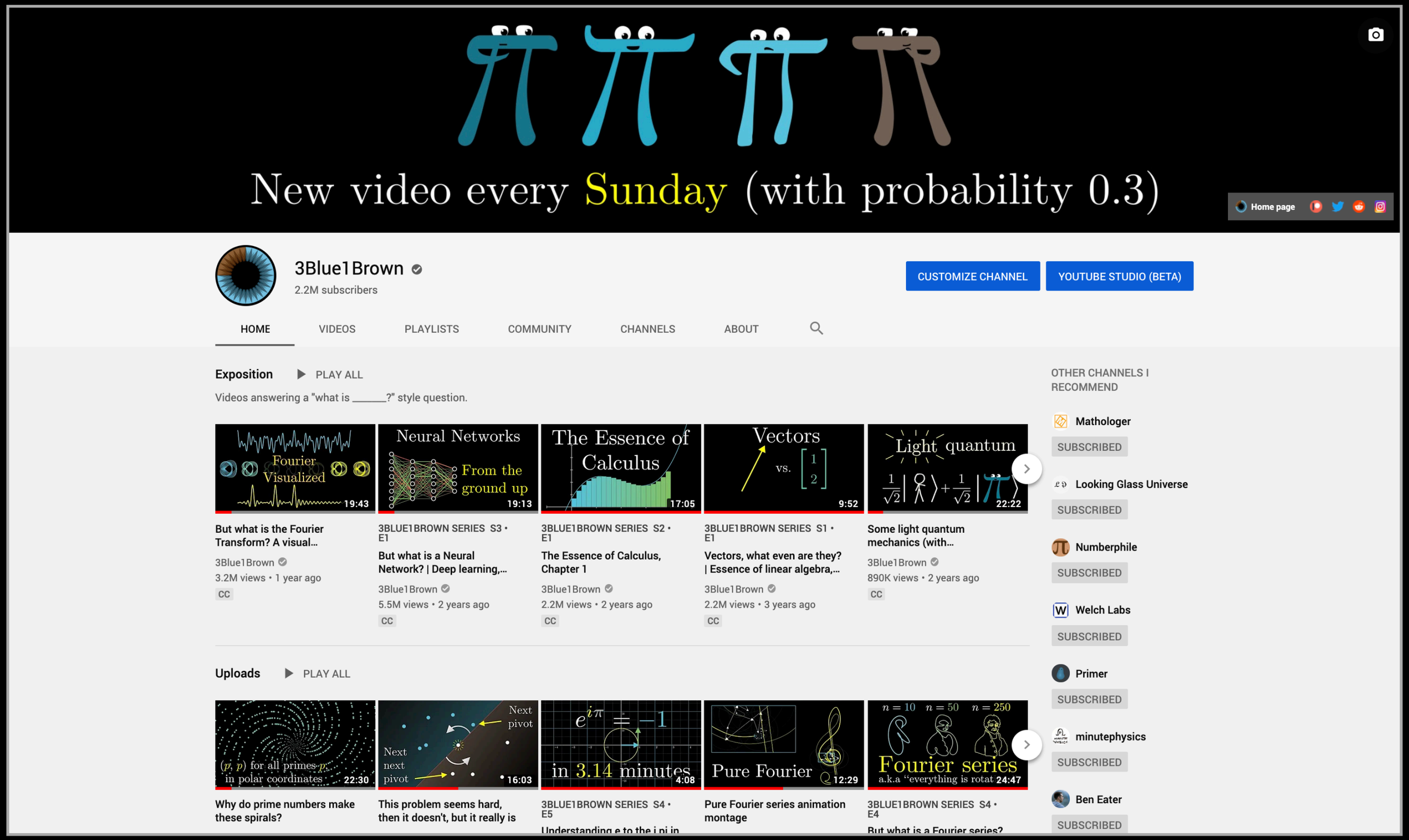
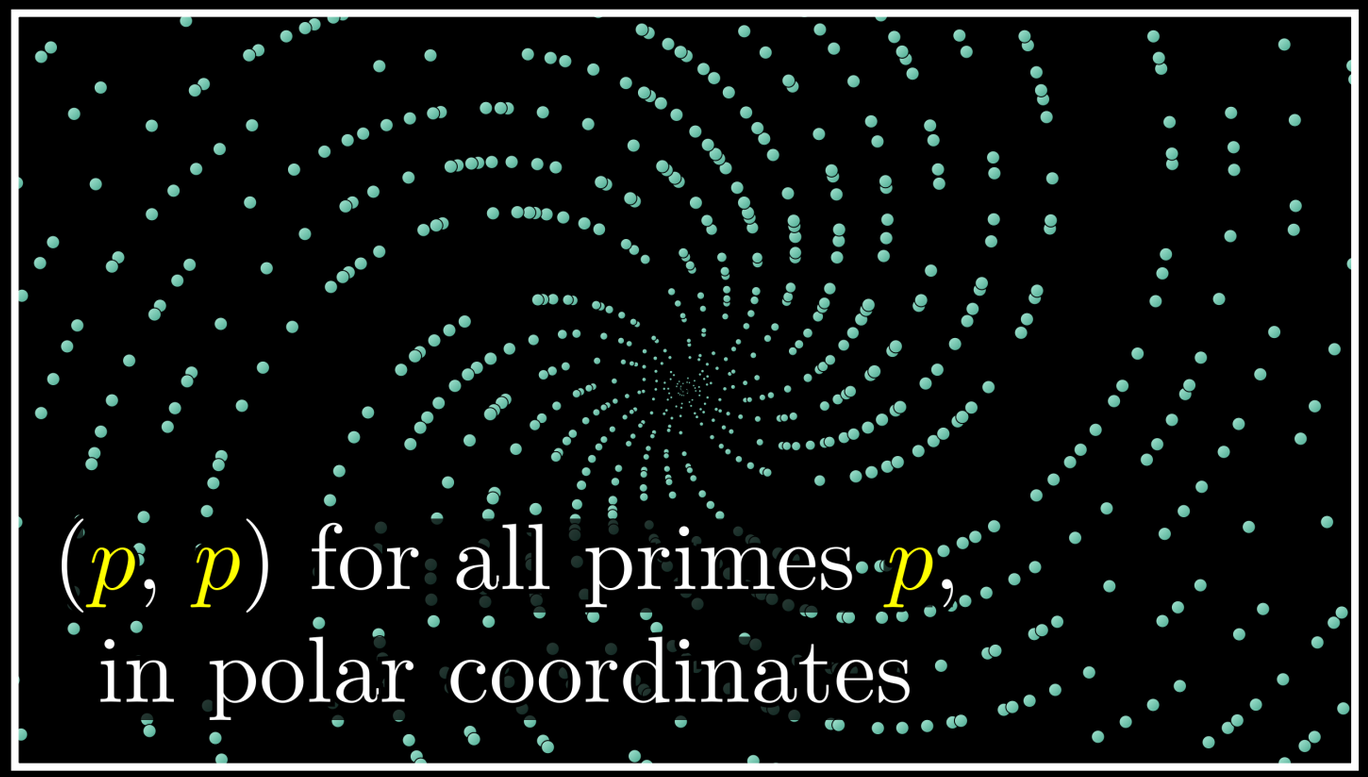
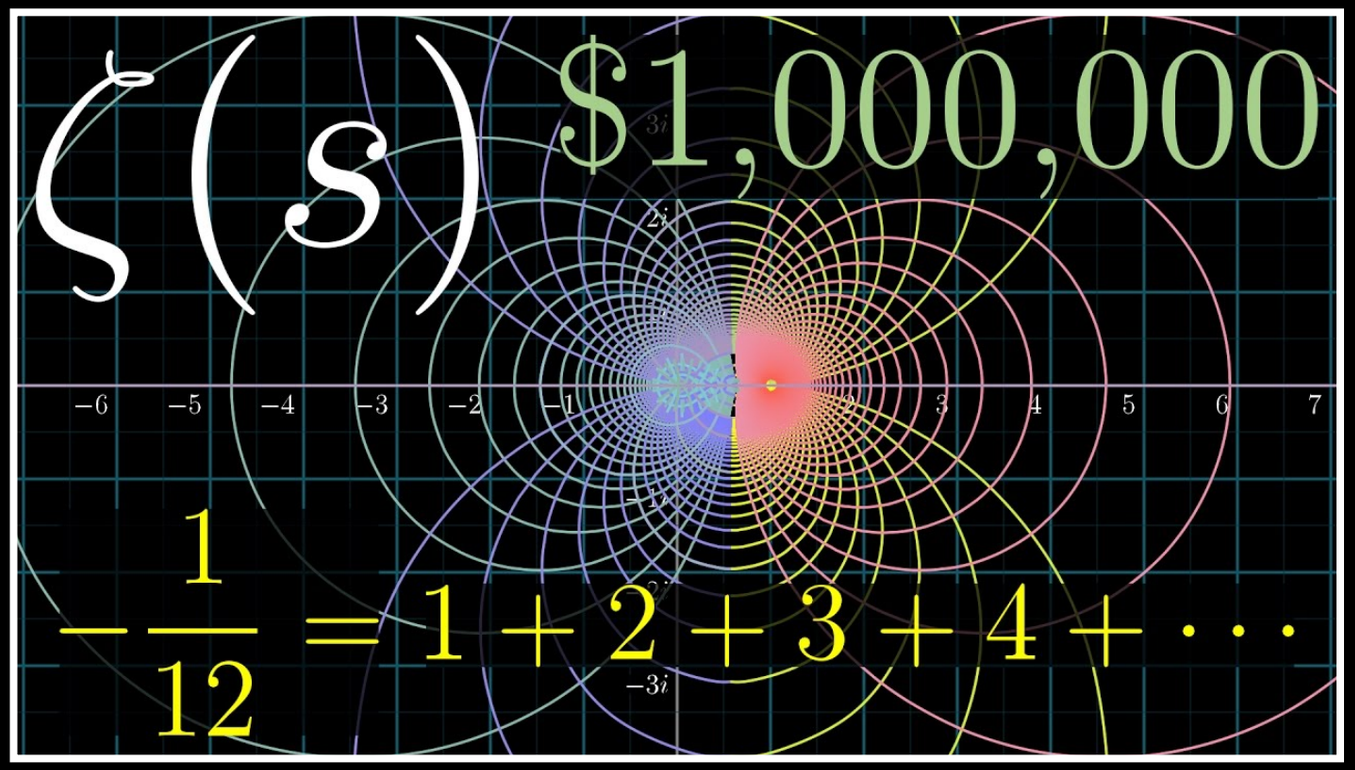
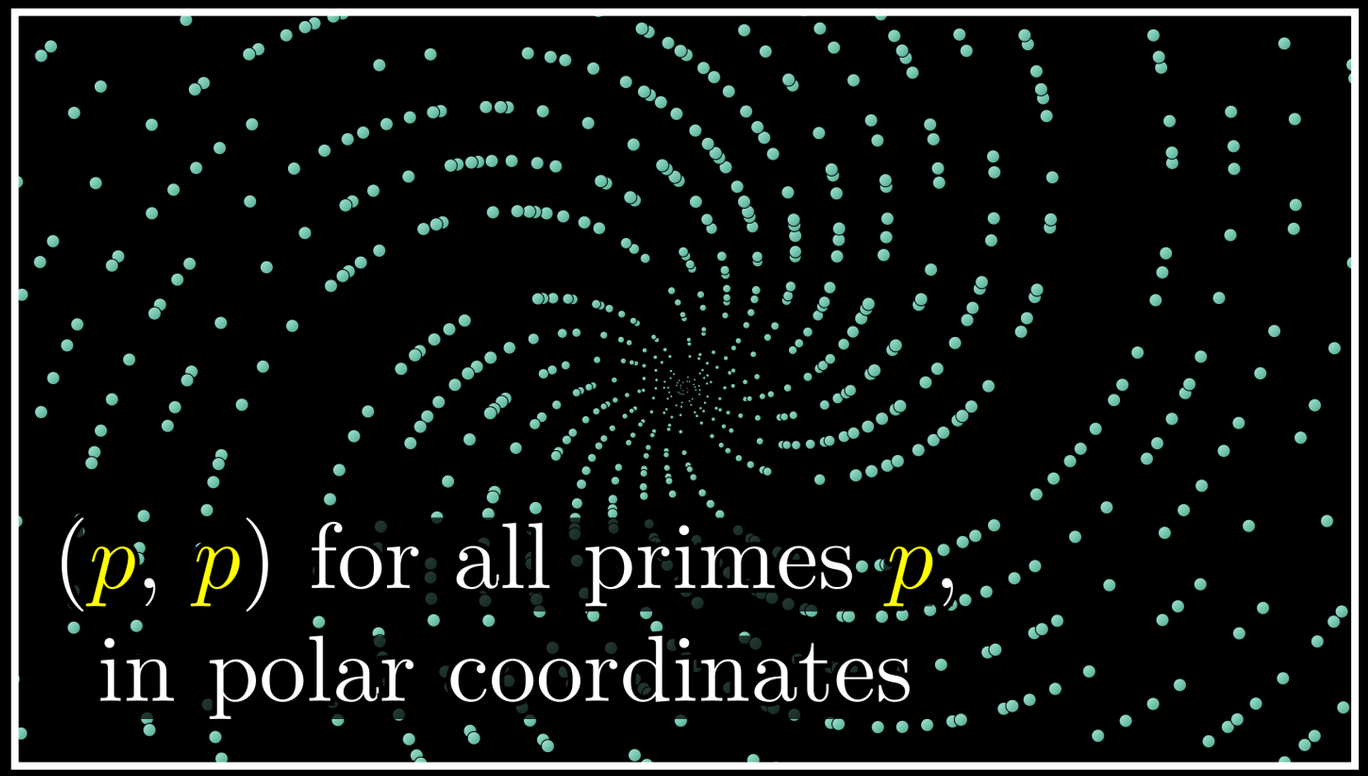
Grant Sanderson
3blue1brown

YouTube Channel

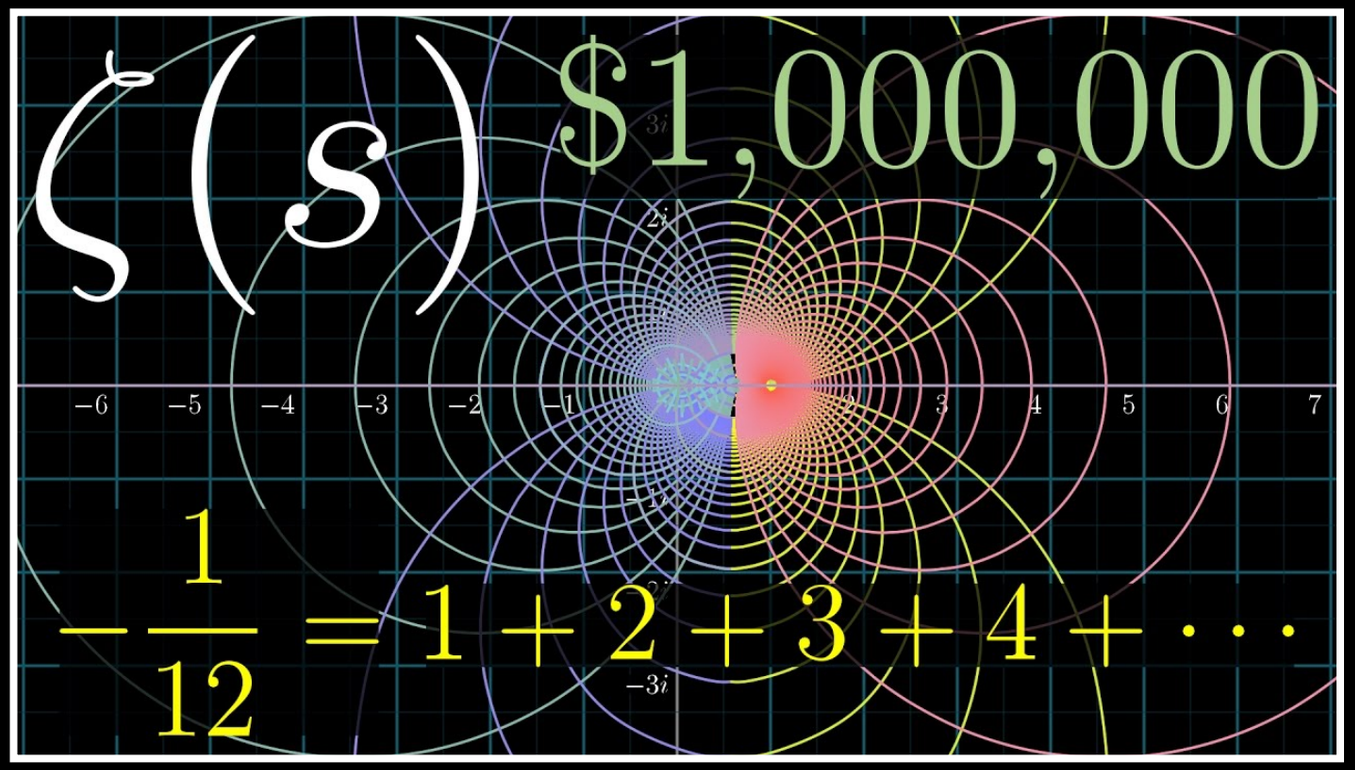


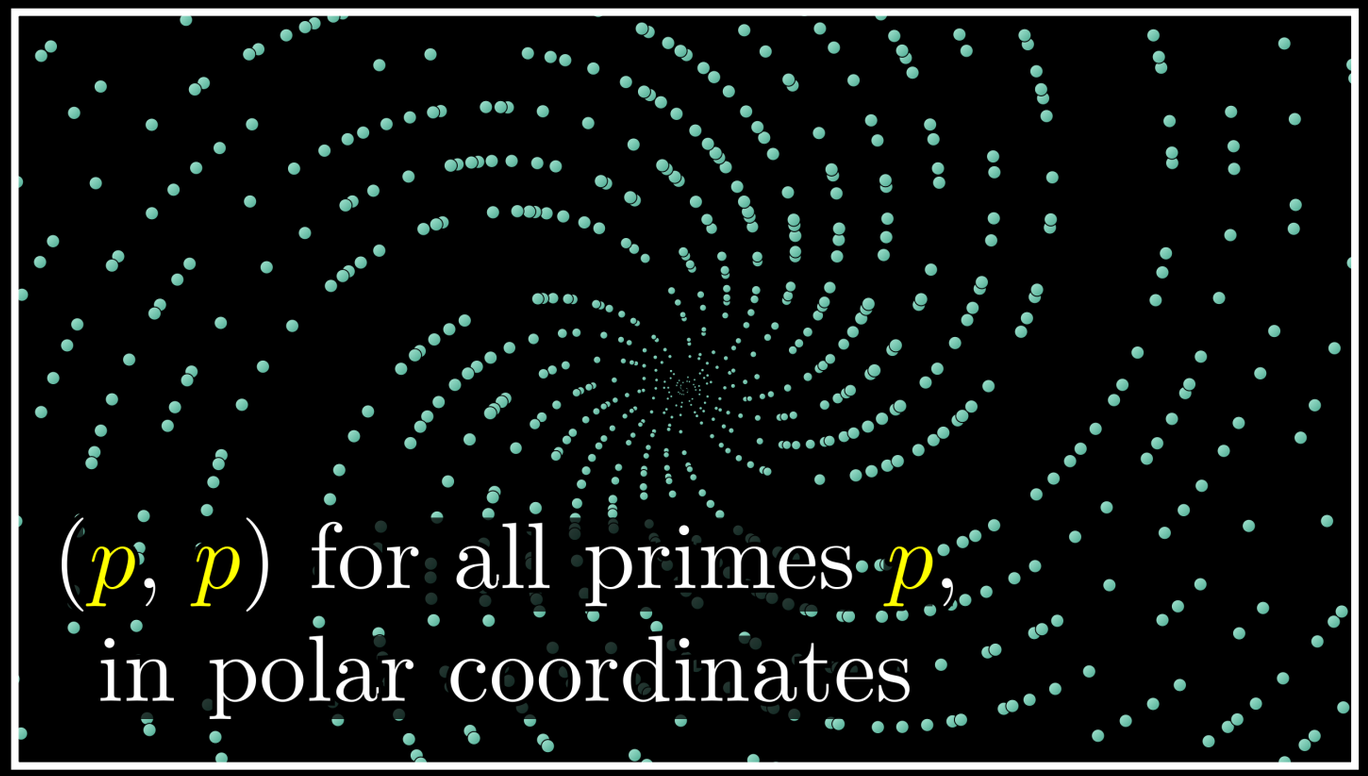

(p, p) for all primes p,
in polar coordinates


$\zeta(s)$
$1,000,000
$-\frac{1}{12} = 1 + 2 + 3 + 4 + \cdots$

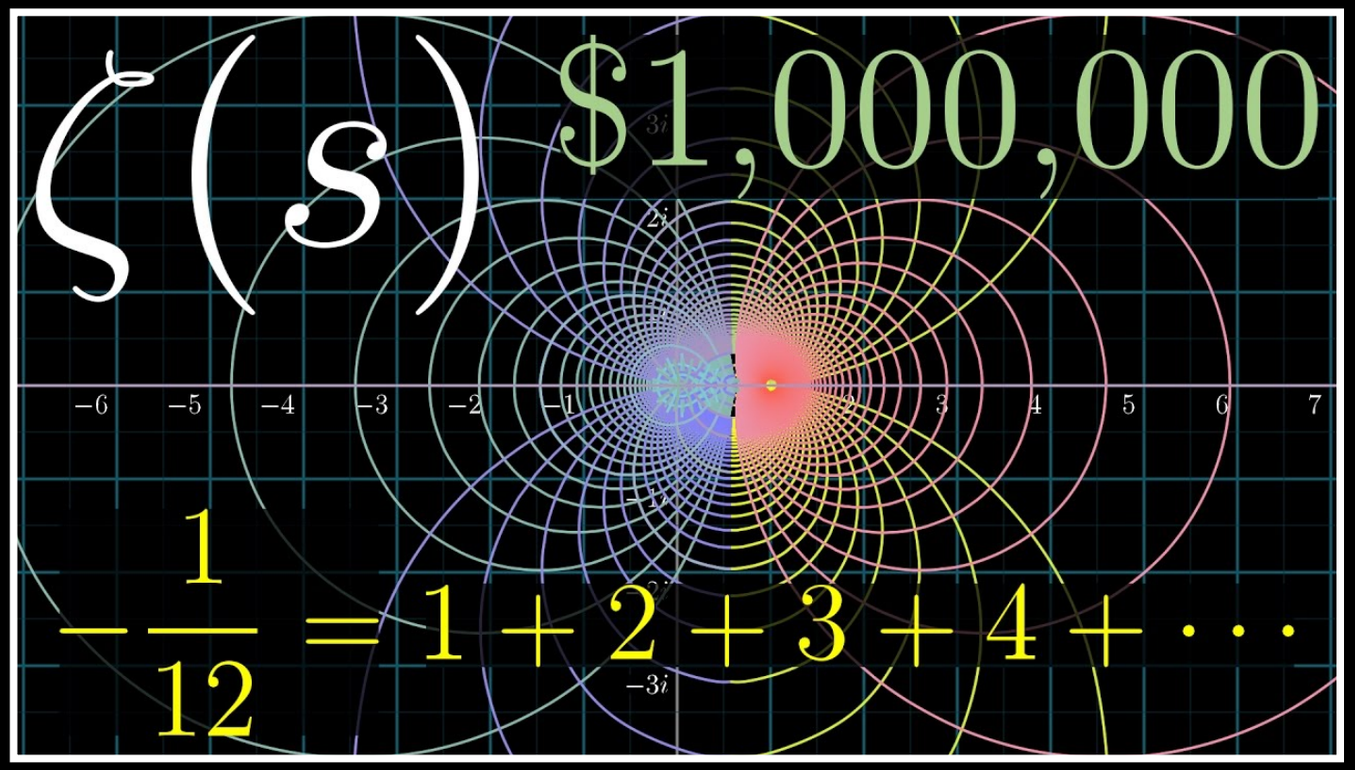


Use visuals to inspire





Use visuals to inspire



Use visuals to elucidate

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Meaning of Rays in Polar Plot of Prime Numbers

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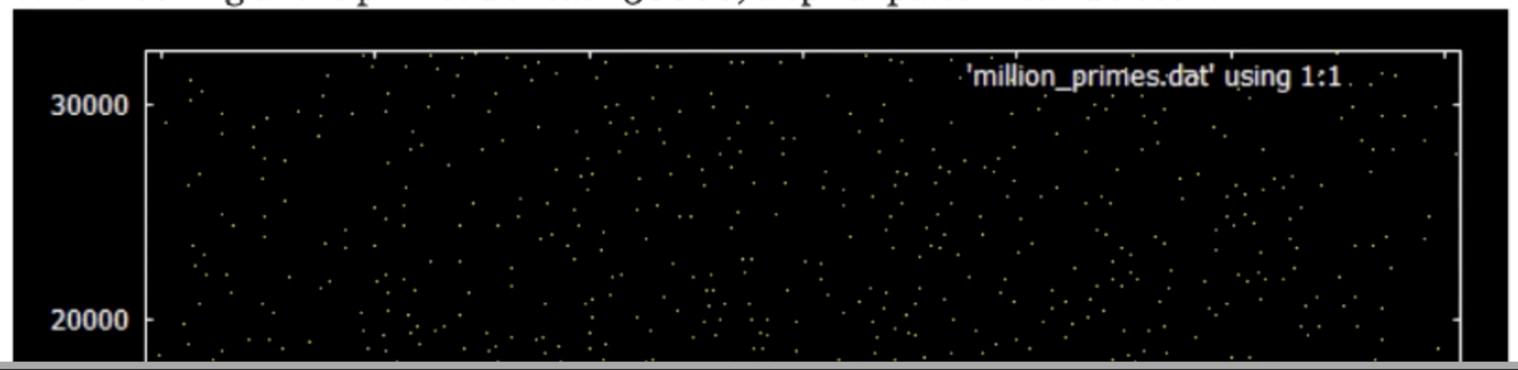
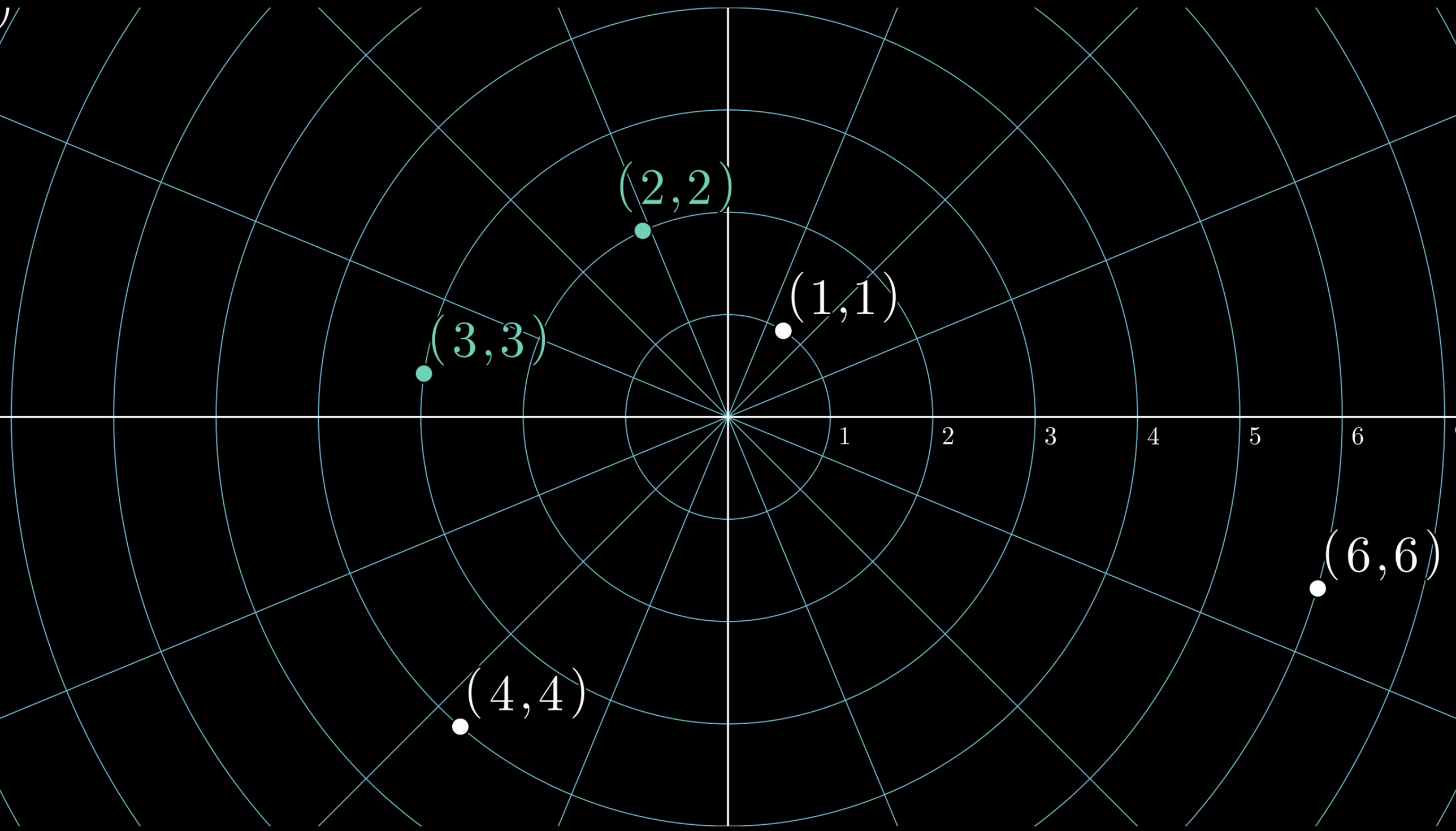
16

11

I recently began experimenting with gnuplot and I quickly made an interesting discovery. I plotted all of the prime numbers beneath 1 million in polar coordinates such that for every prime p, $(r, \theta) = (p, p)$. I was not expecting anything in particular, I was simply trying it out. The results are fascinating.

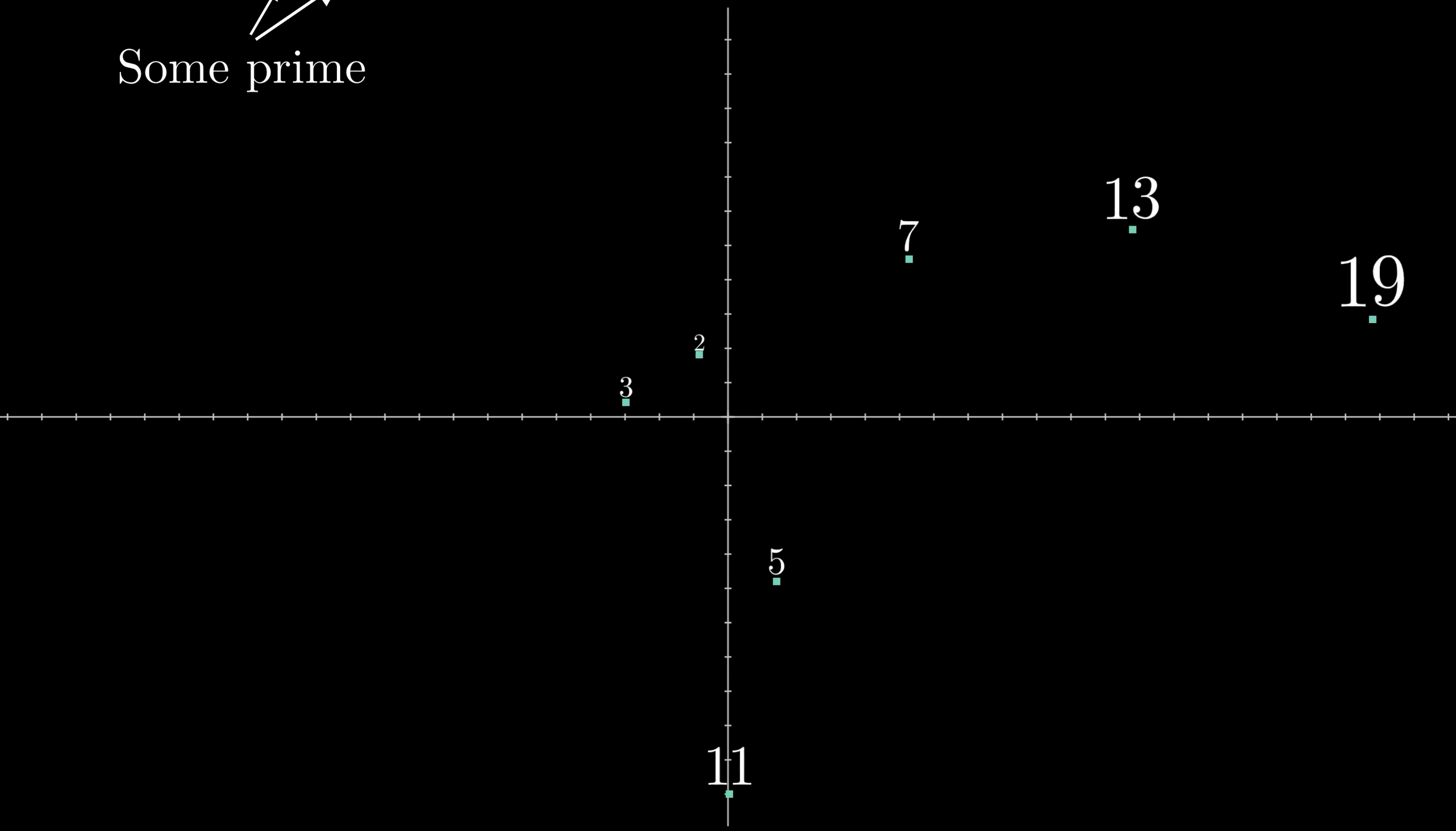
When looking at the primes beneath 30000, a spiral pattern can be seen:

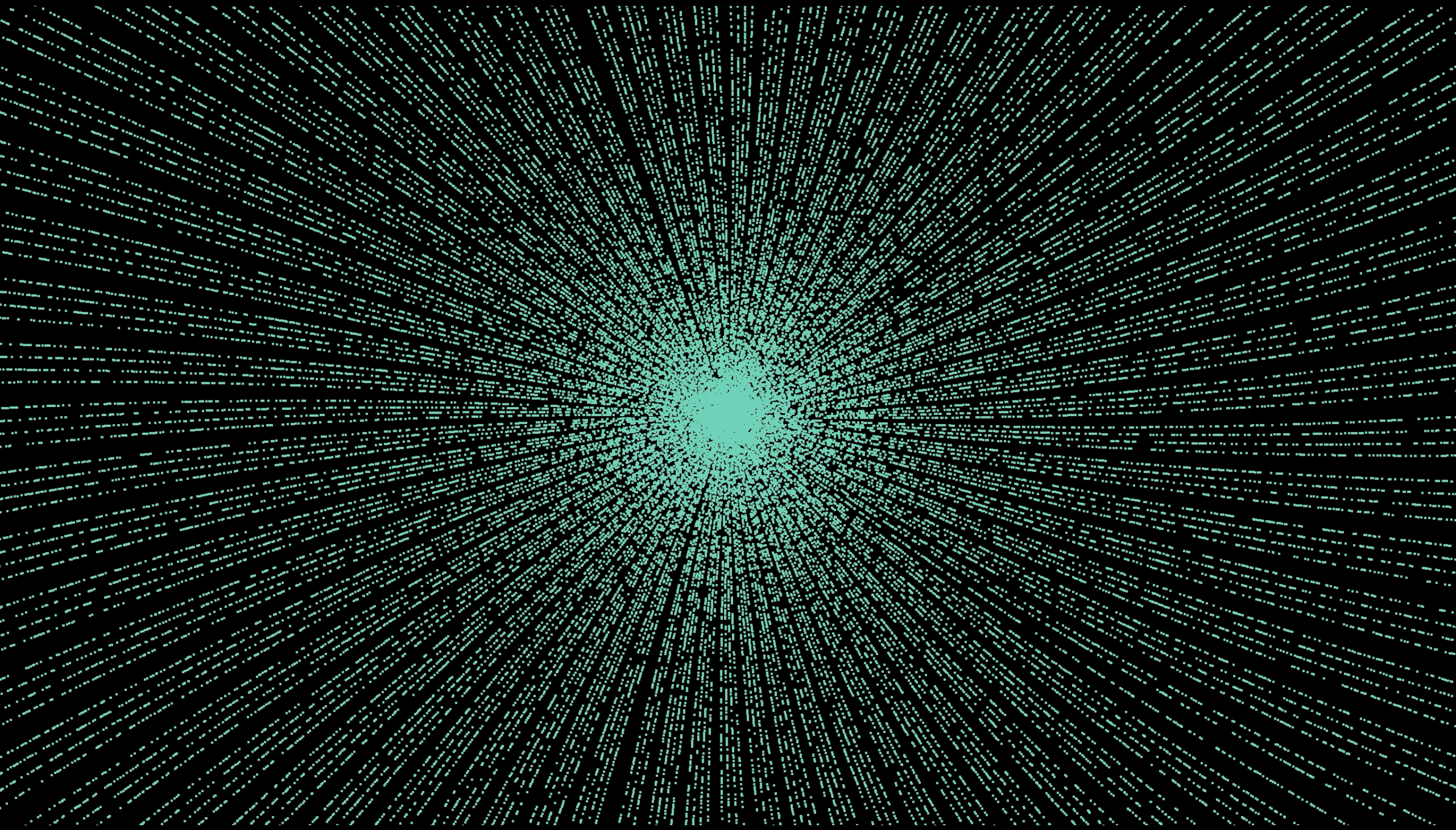



(2,2)
(1,1)
(3,3)
1
2
3
4
5
6
(6,6)
(4,4)

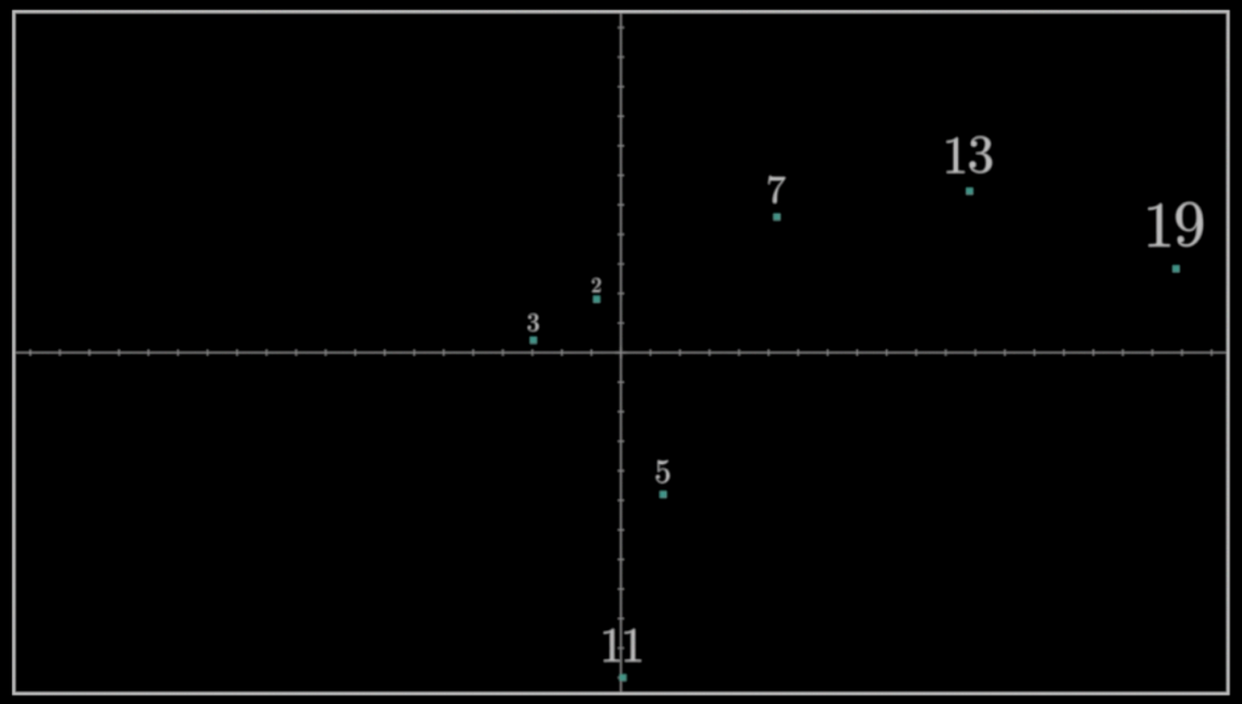
$$(r, \theta) = (p, p)$$

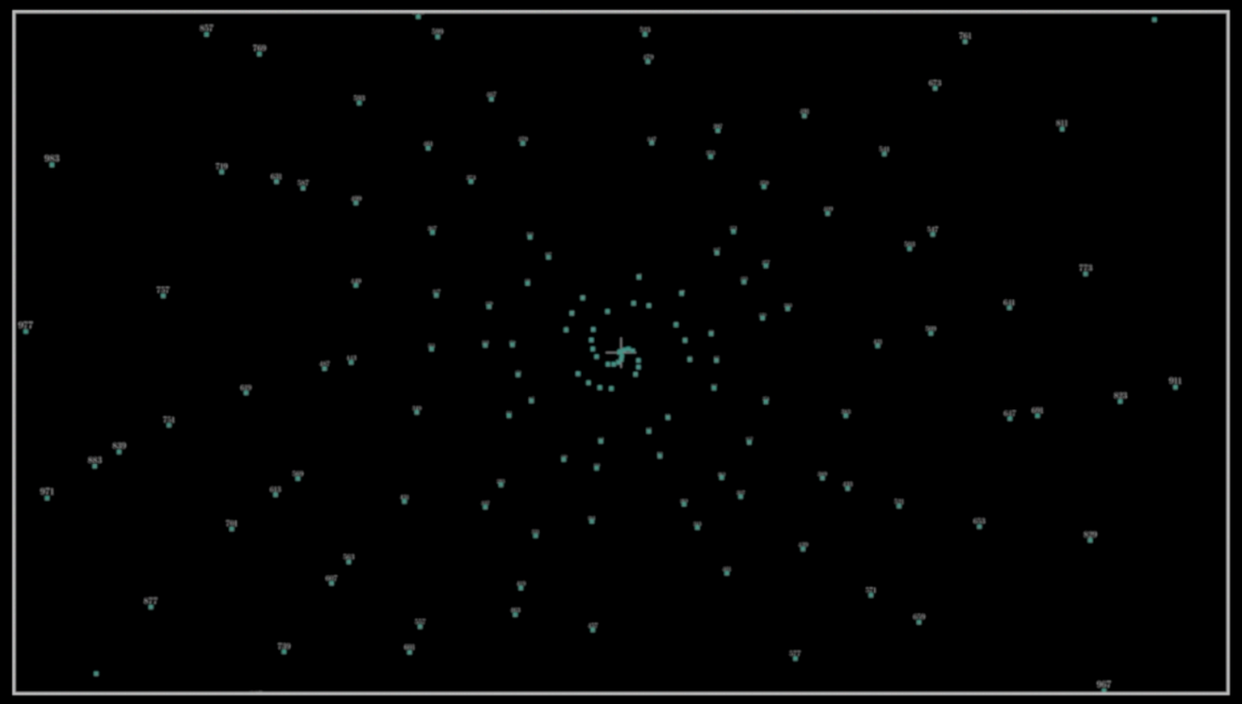
Some prime

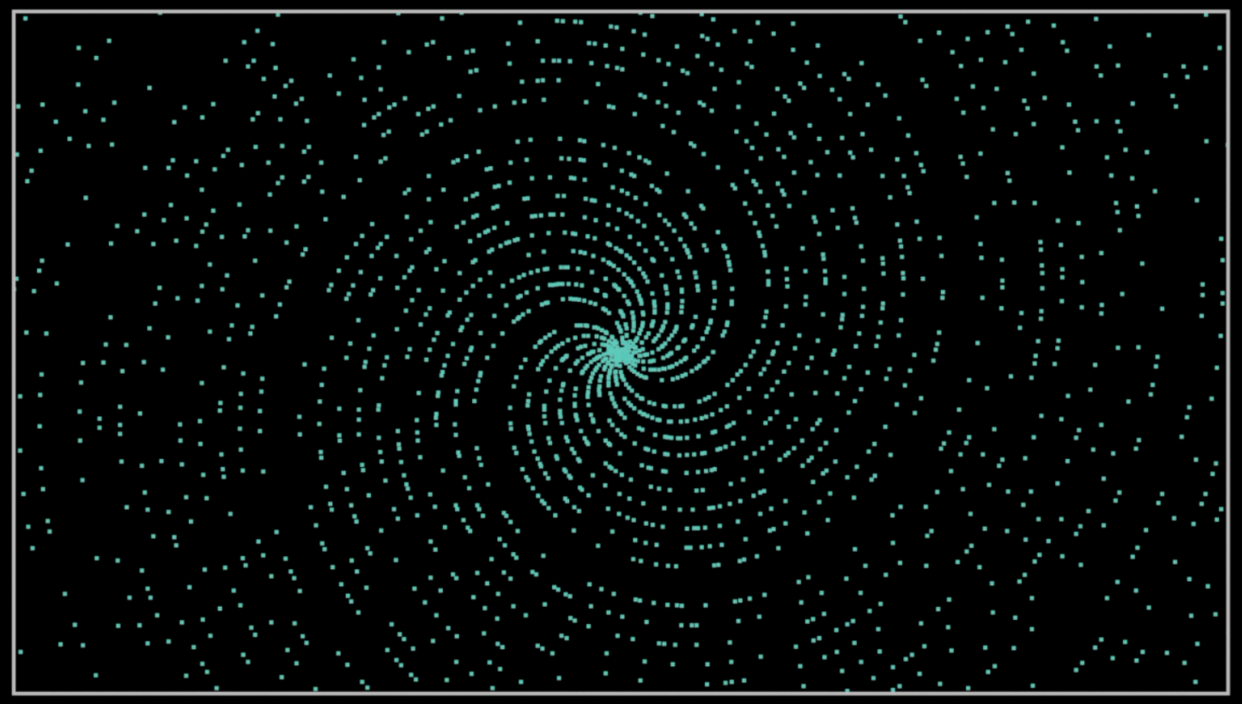


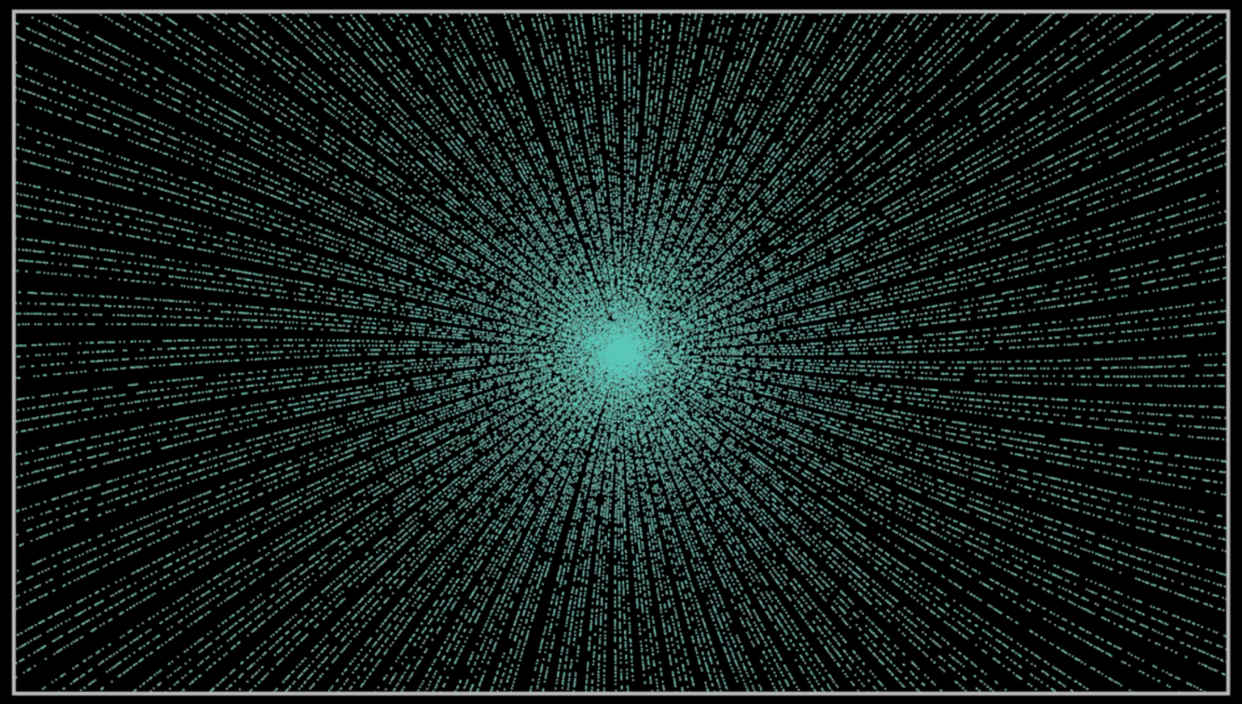


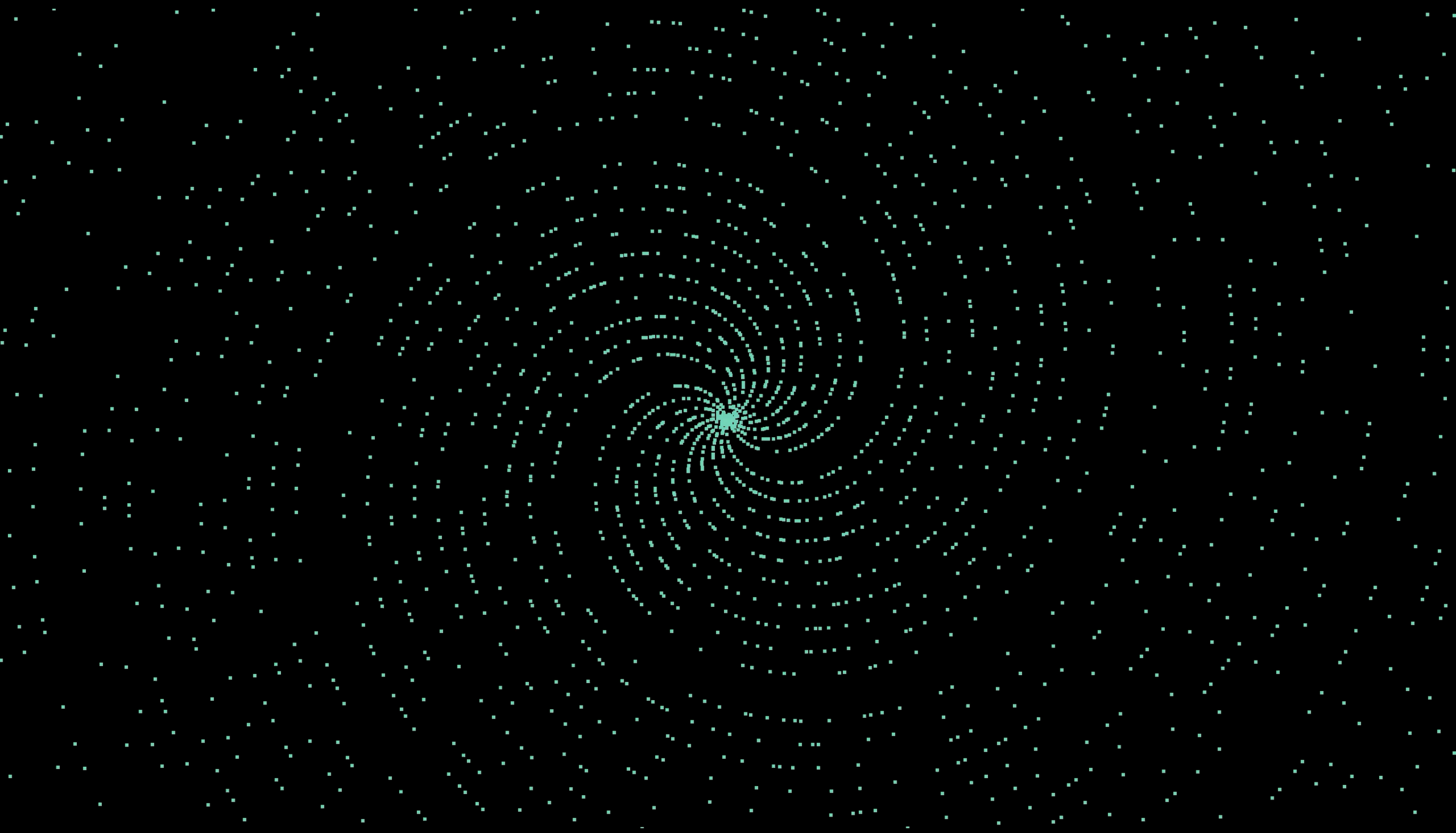
WHY?!

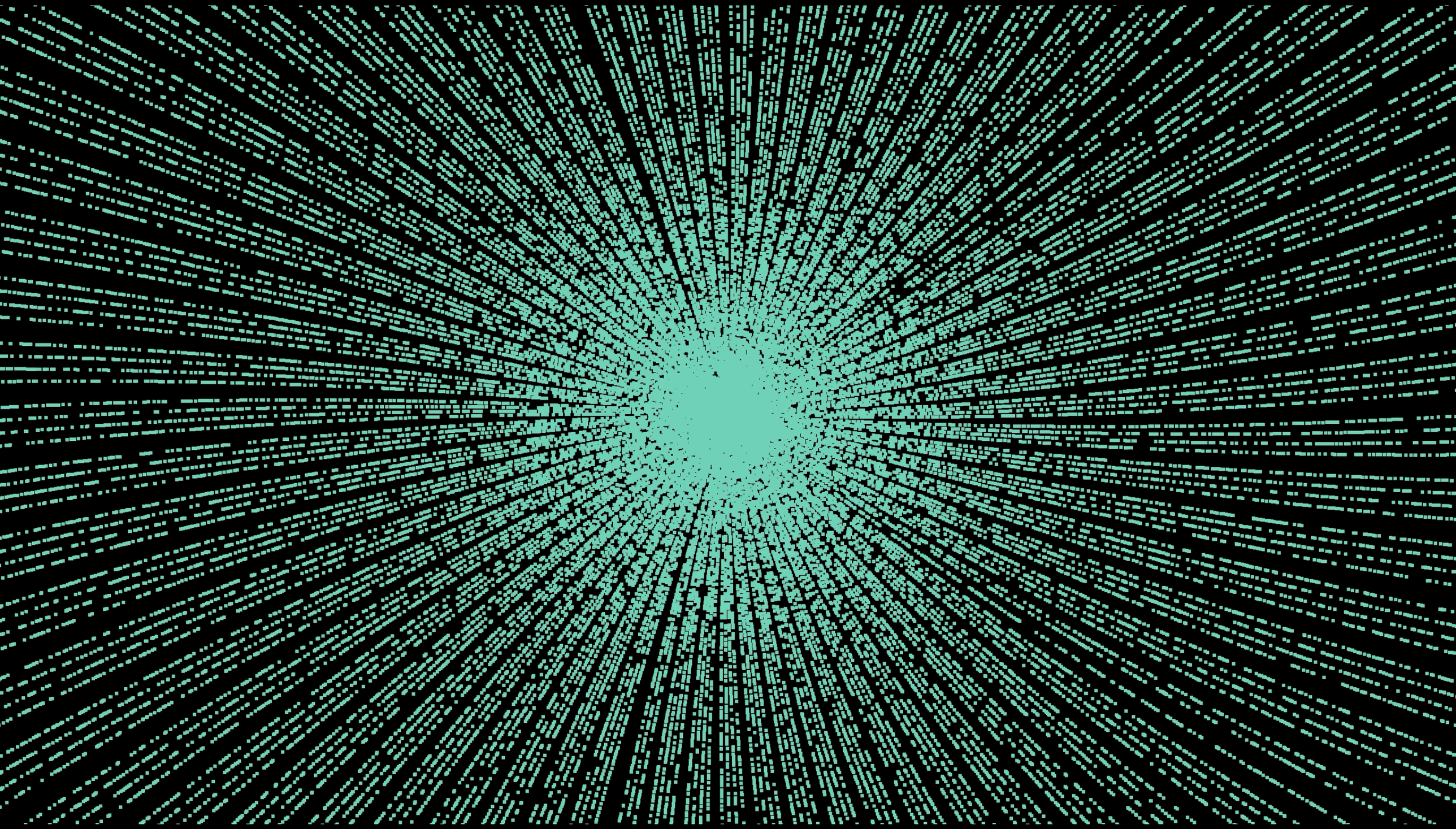




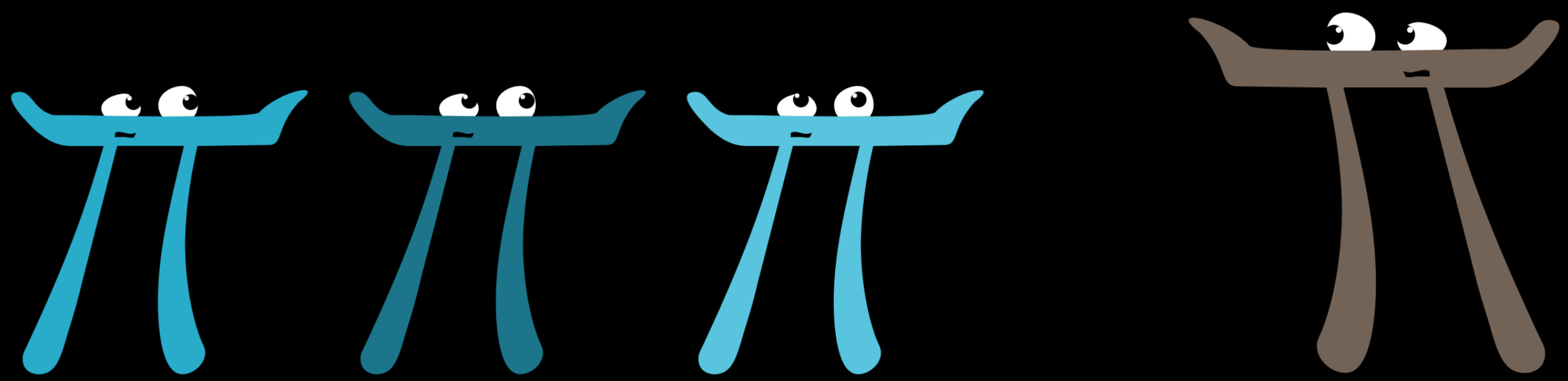


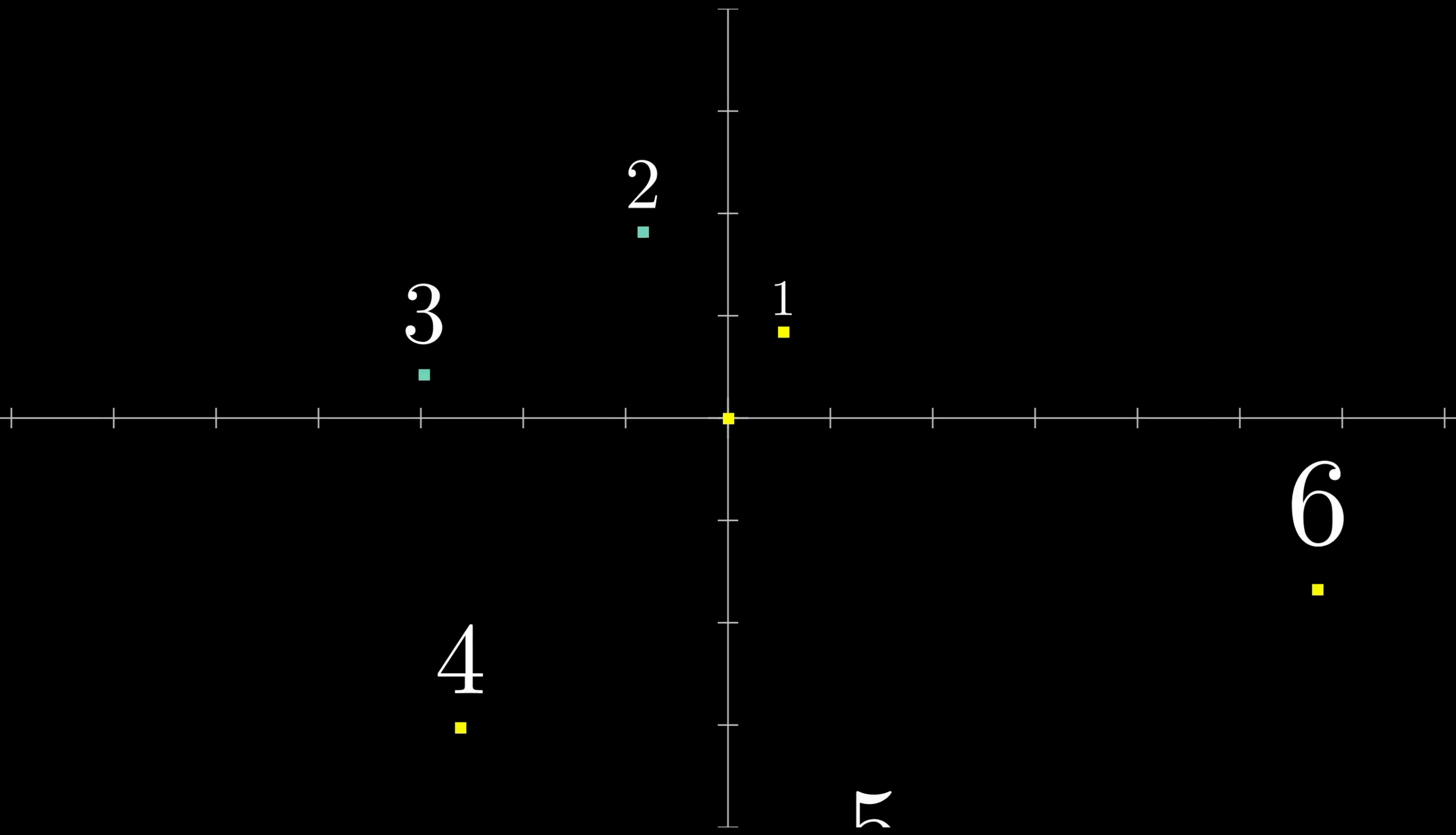






$$2,\ 3,\ 5,\ 7,\ 11,\ \ldots \xrightarrow{???} 20,\ 280$$

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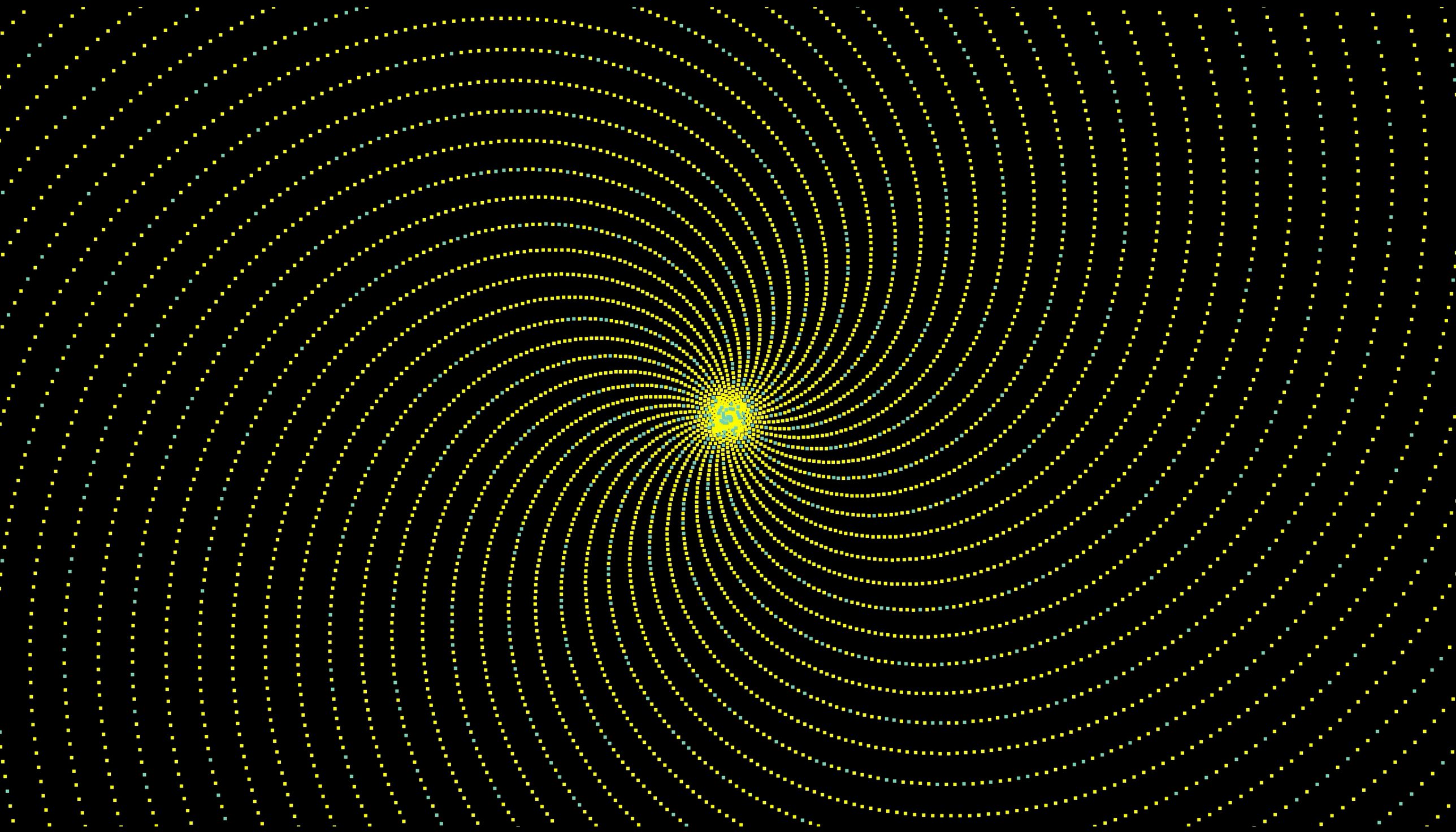
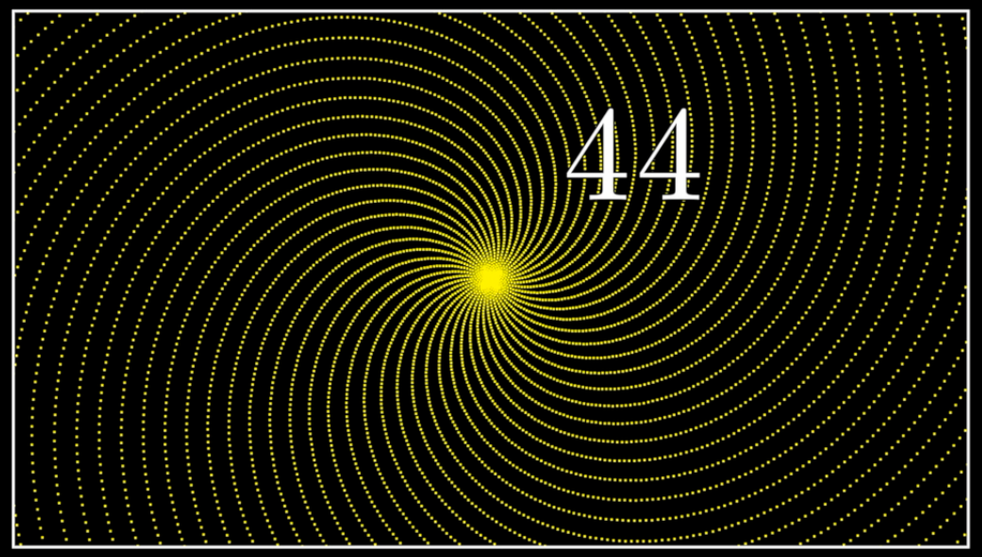
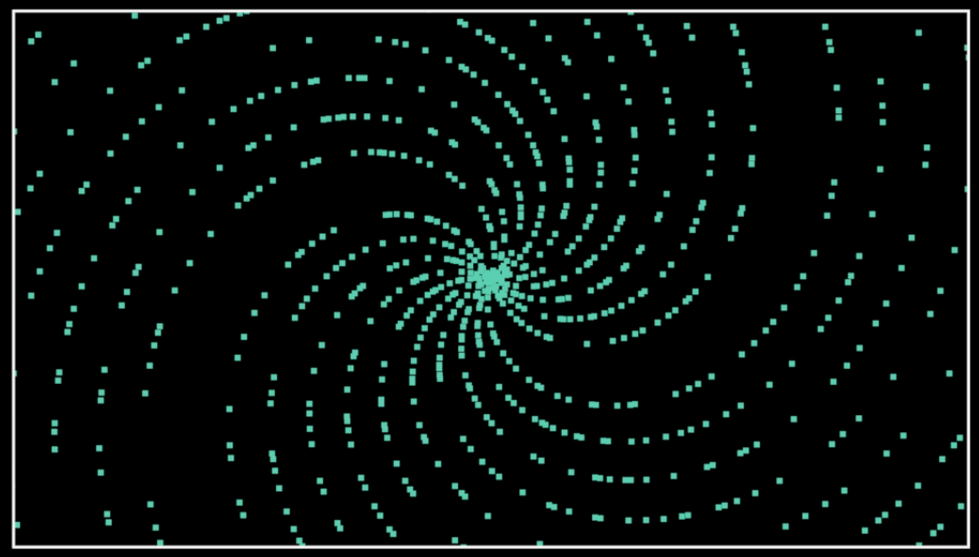


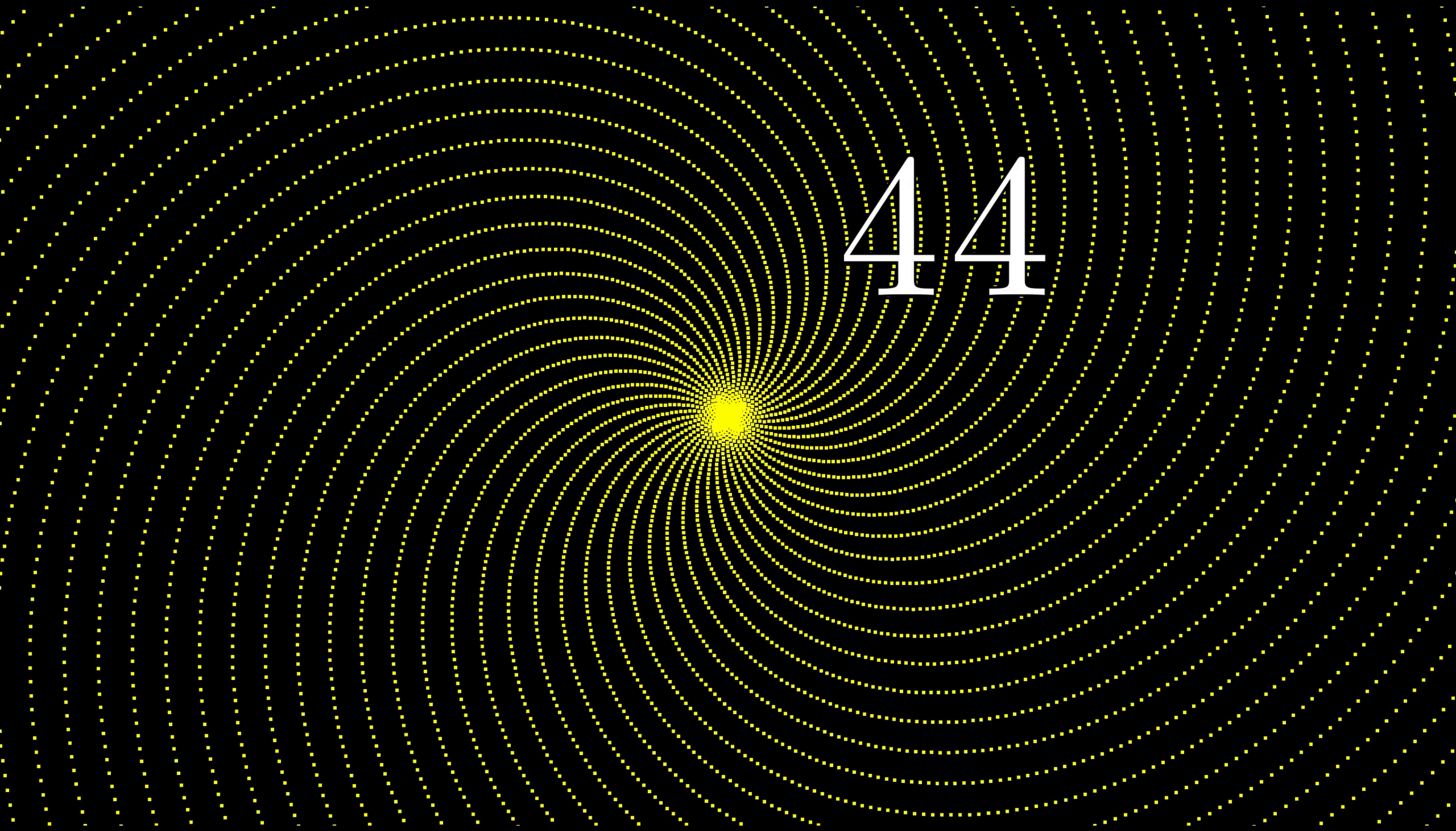
~~Why do primes cause spirals?~~

Where do the spirals come from?

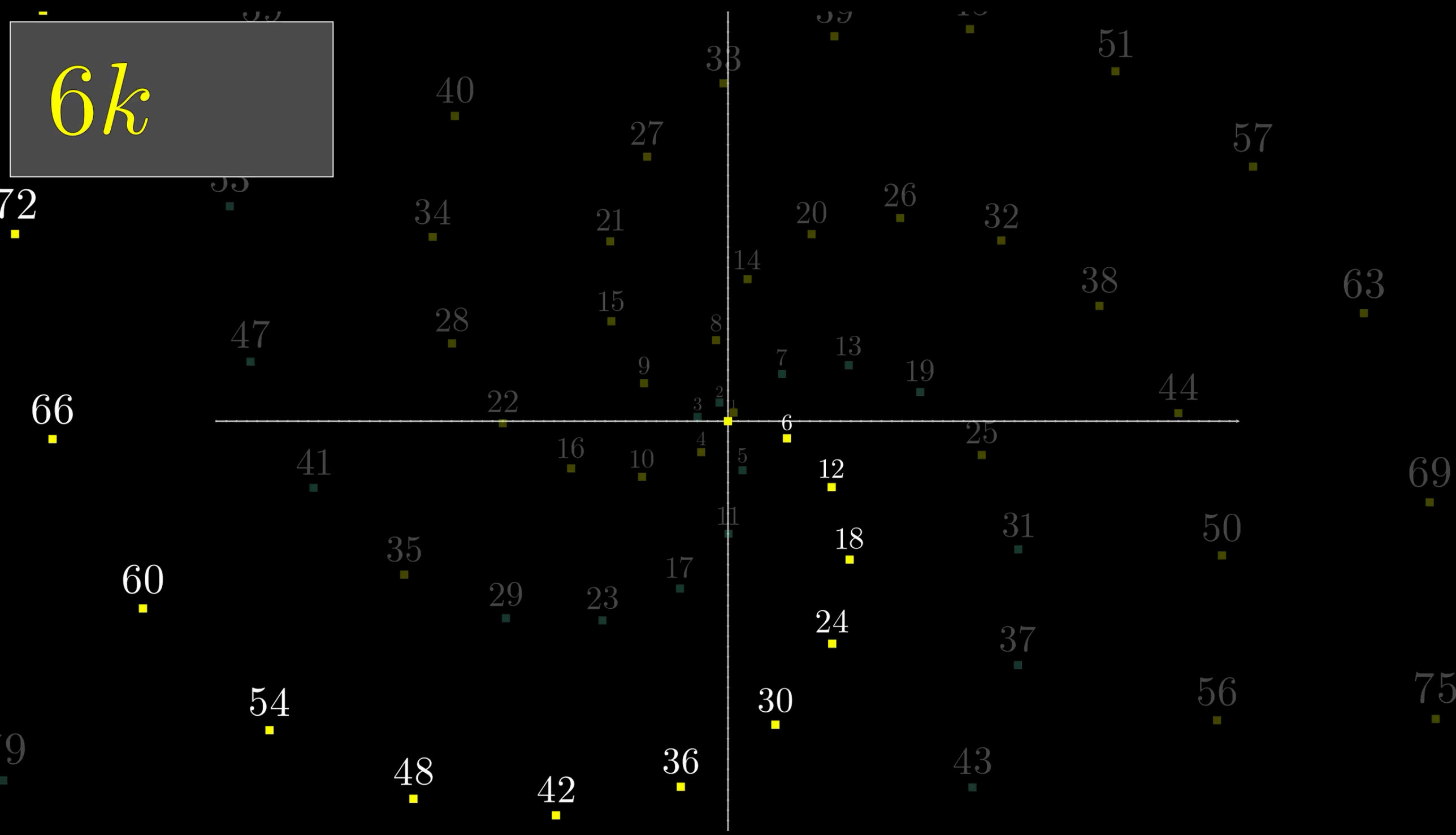


What happens when filtering to primes?




44

$6k$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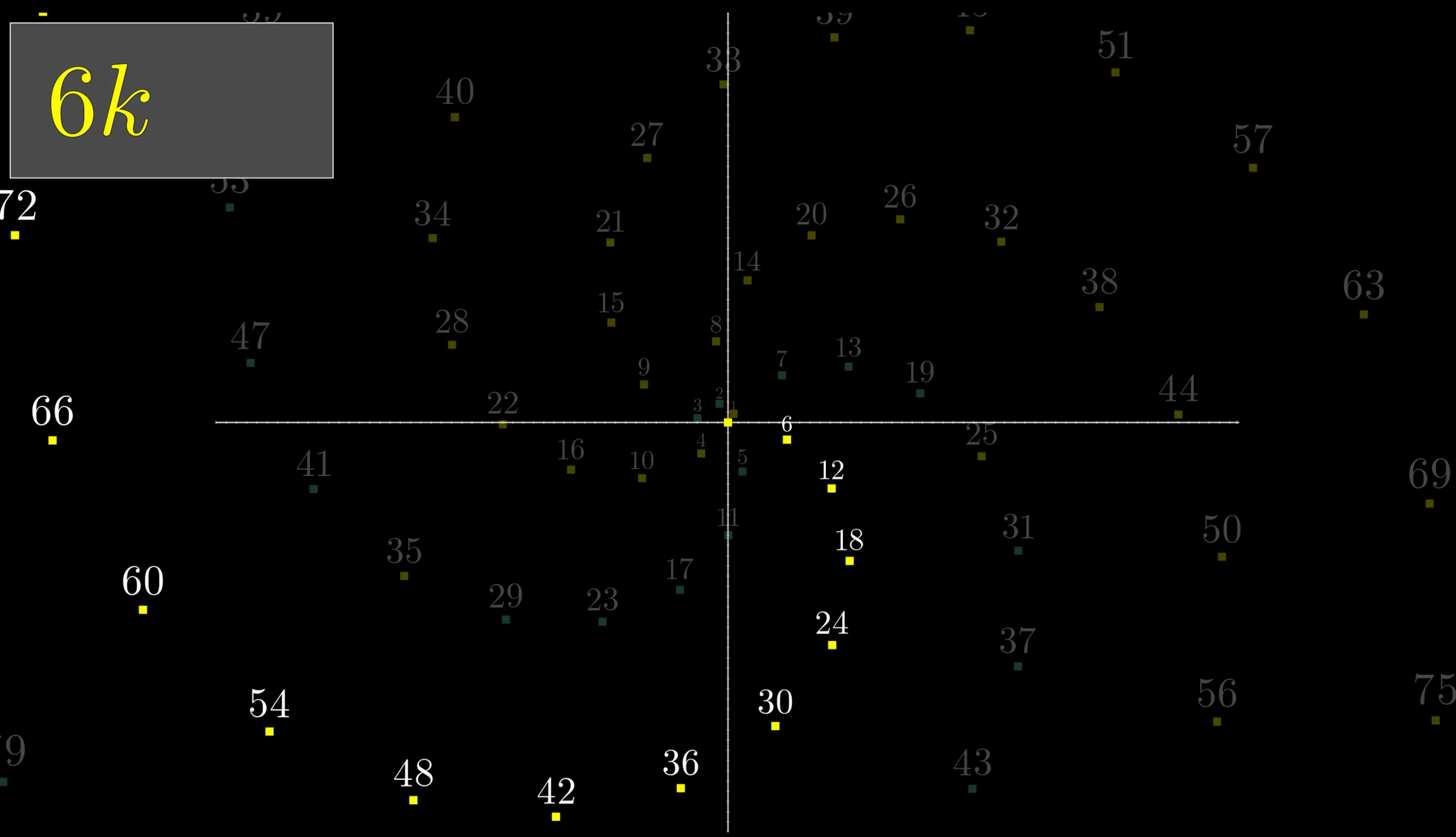
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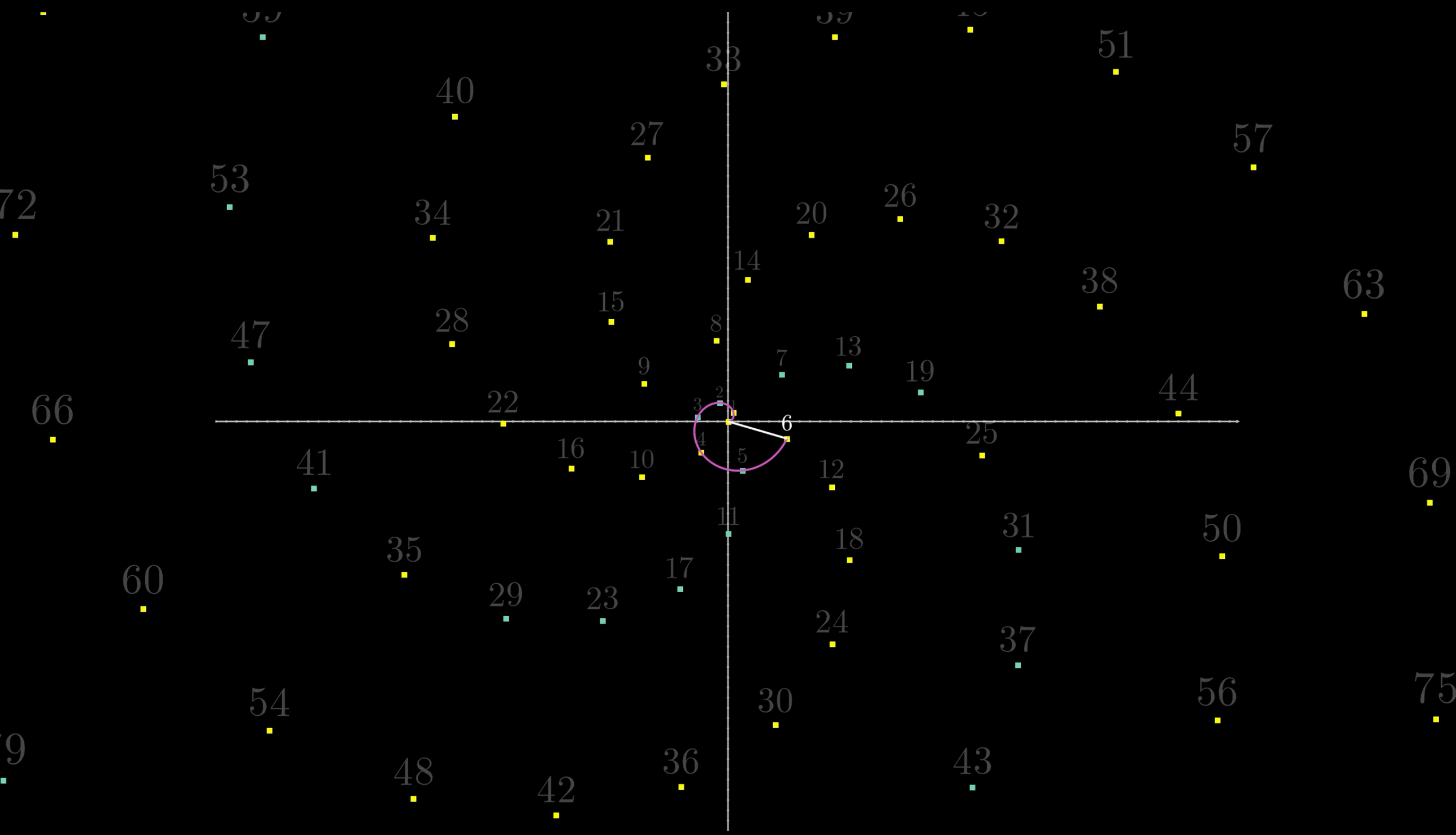
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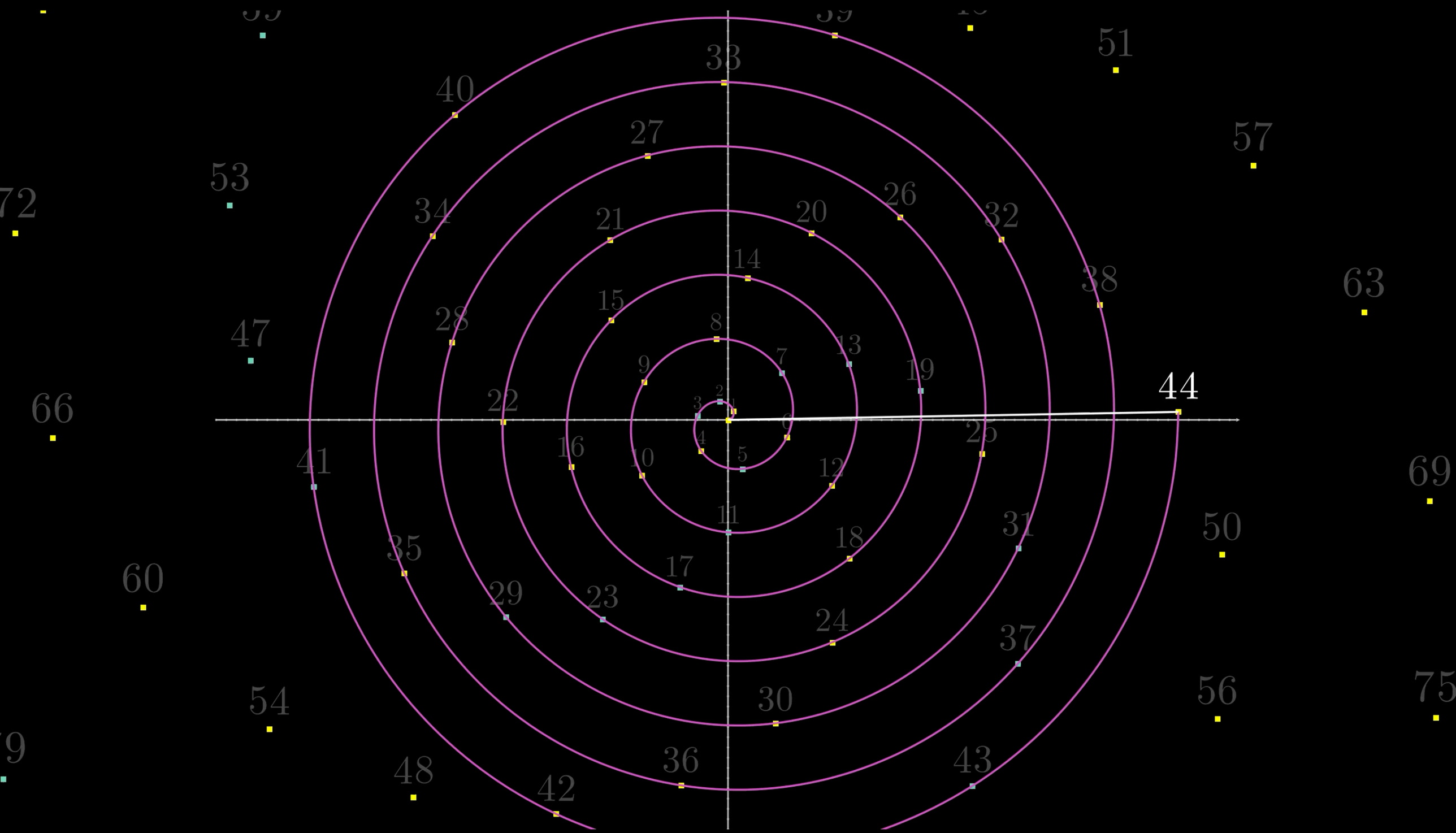
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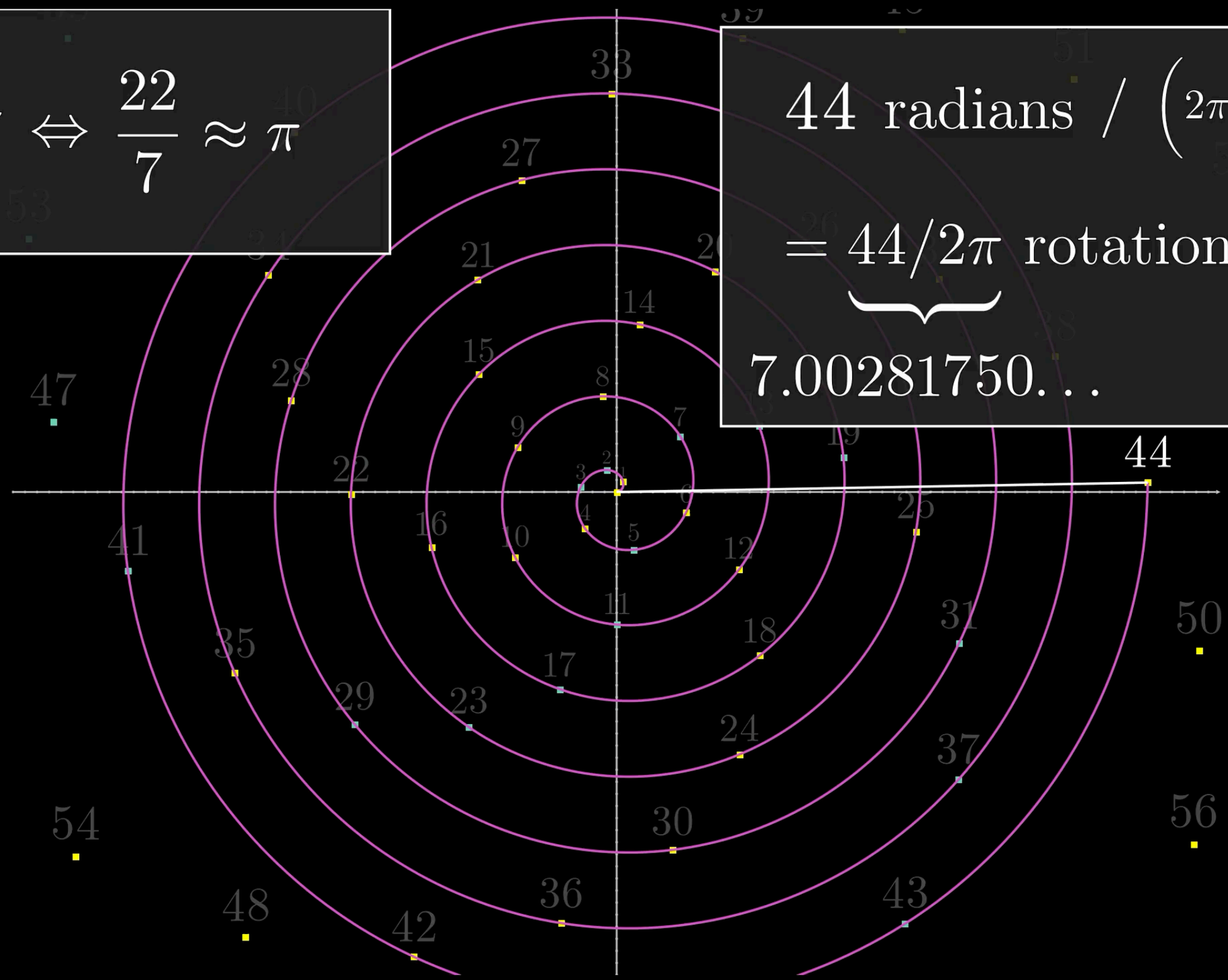
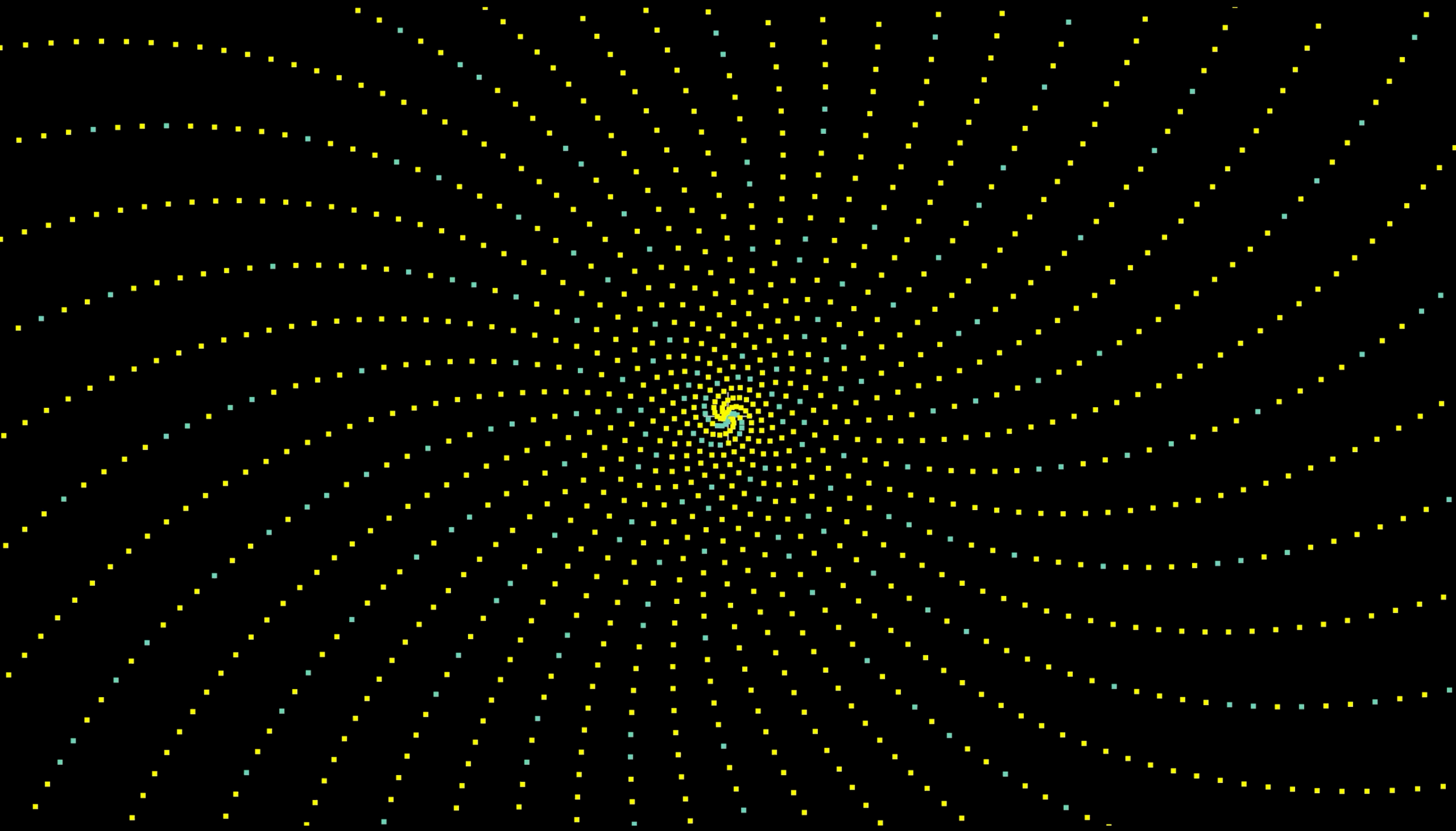
42

$$\frac{44}{2\pi} \approx 7 \Leftrightarrow \frac{22}{7} \approx \pi$$

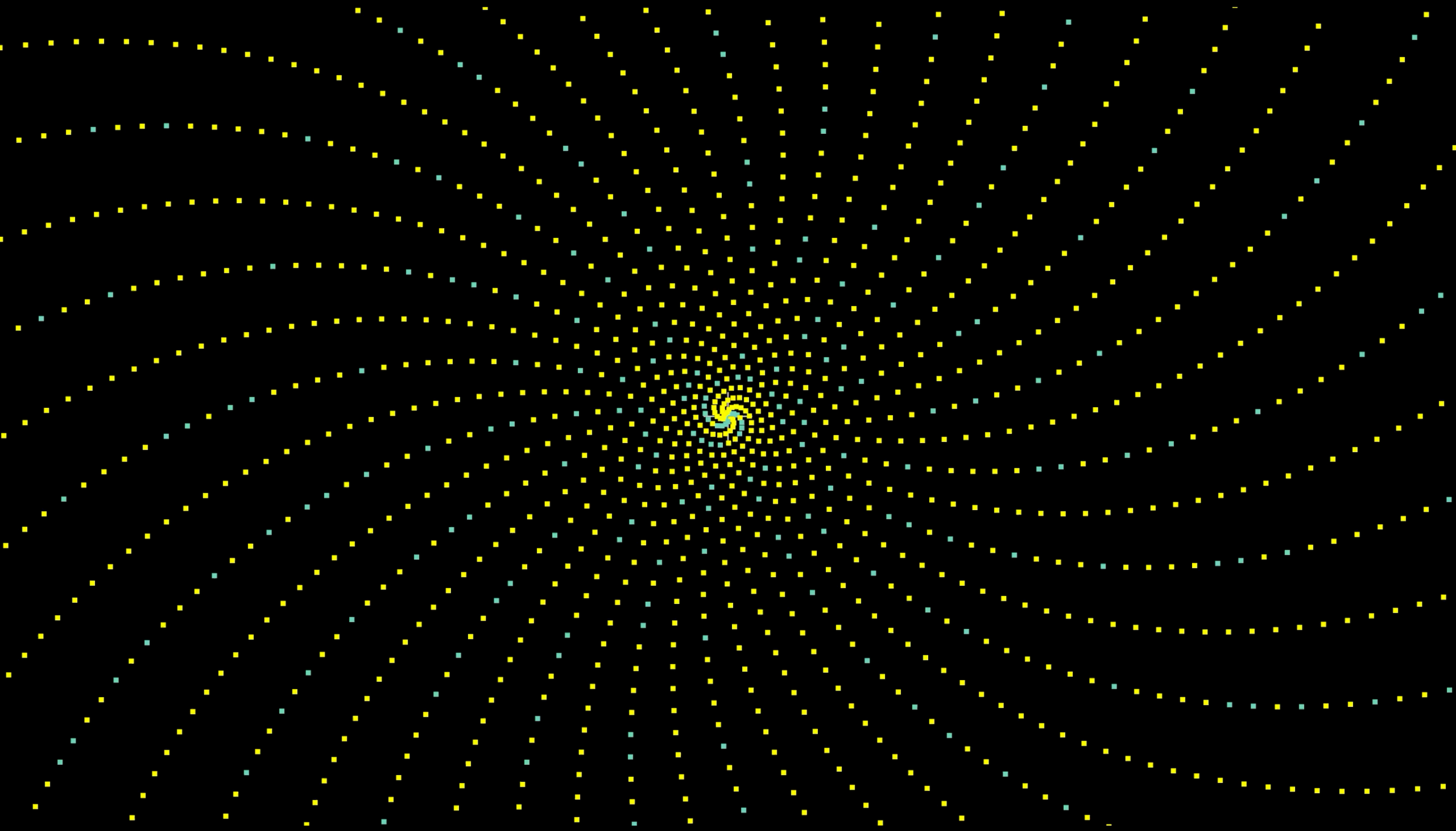
$$44 \text{ radians} \;/\; \left(2\pi \frac{\text{radians}}{\text{rotations}}\right)$$

$$= \underbrace{44/2\pi}_{7.00281750\ldots} \text{ rotations}$$

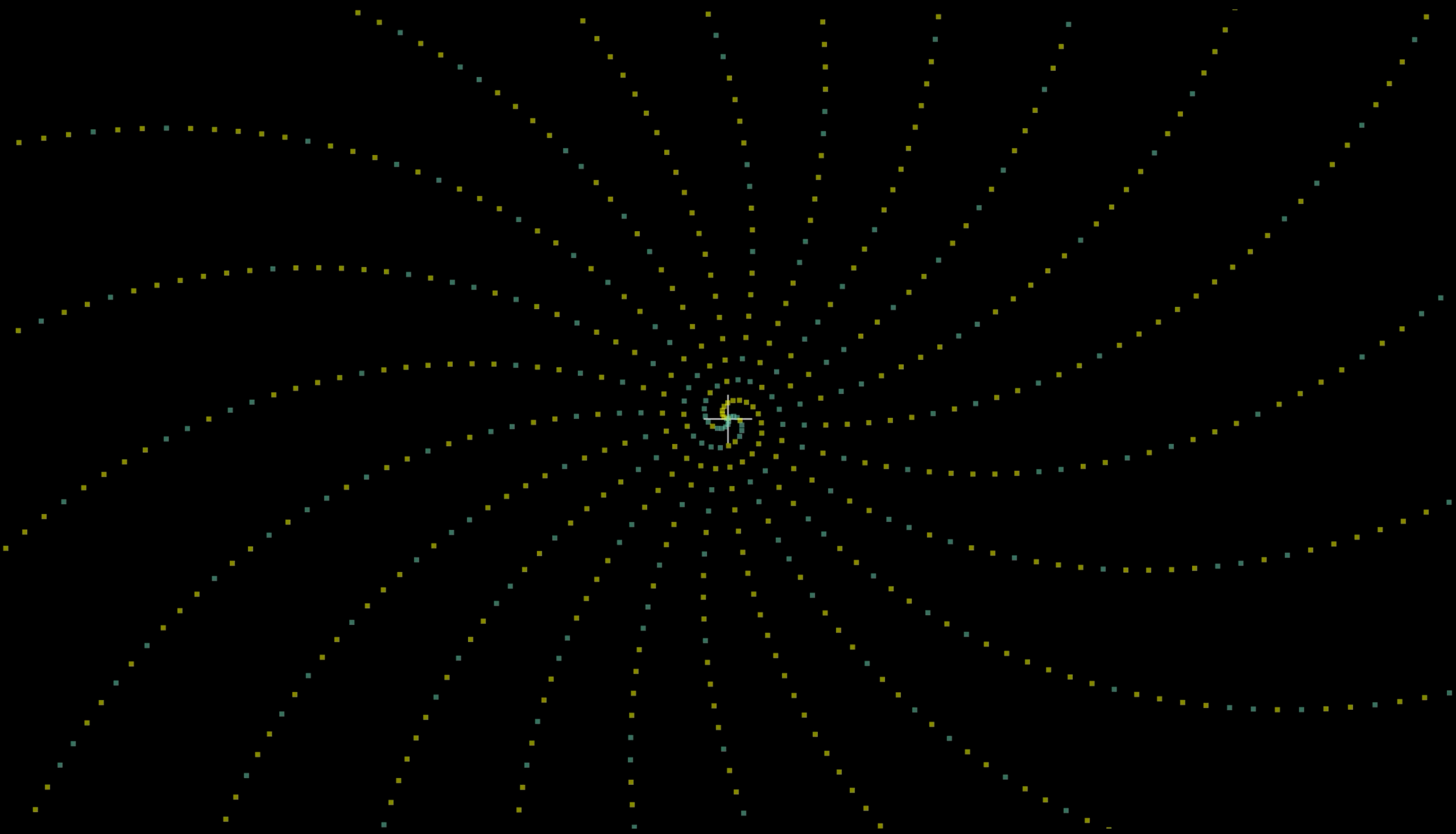


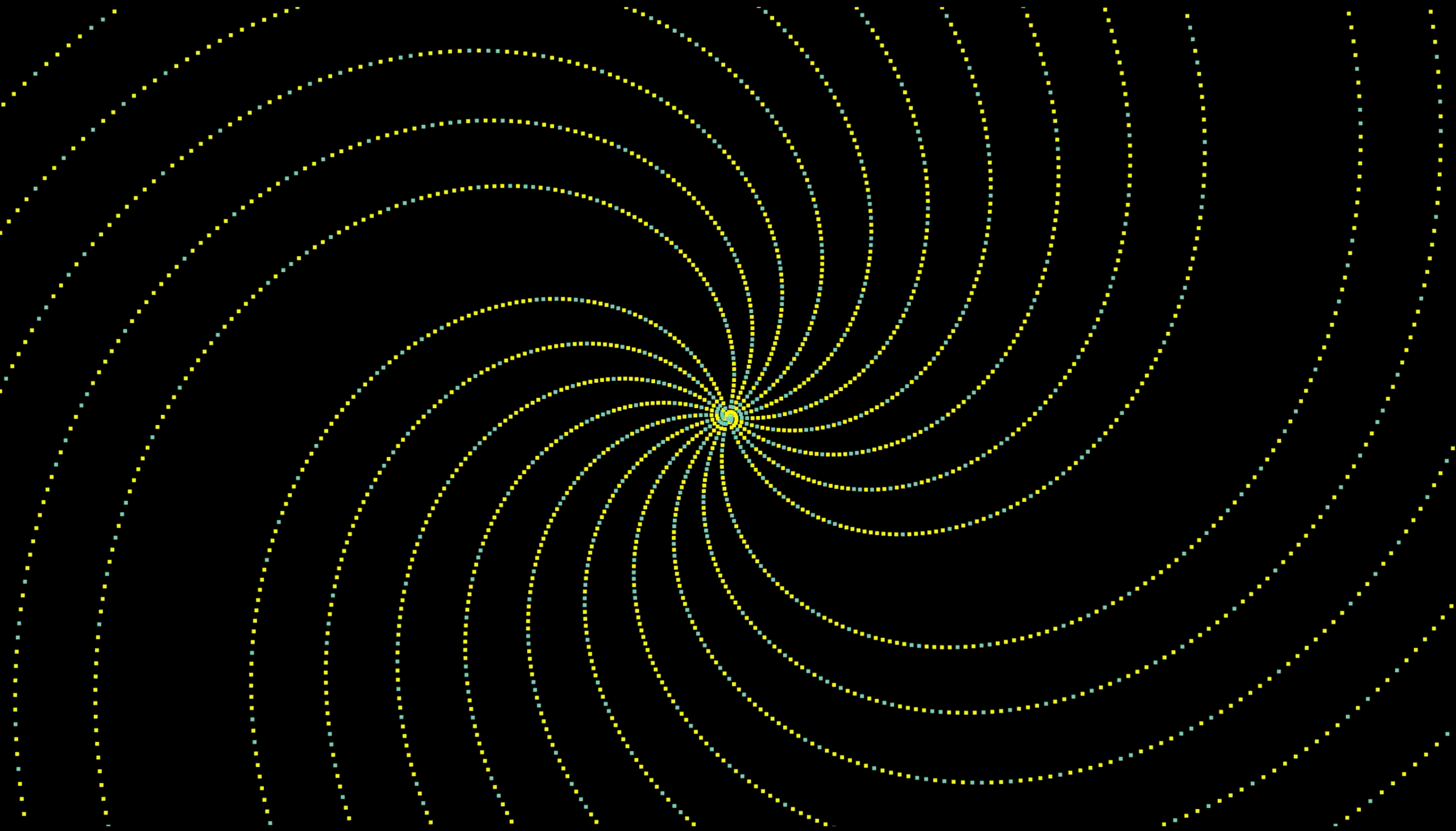


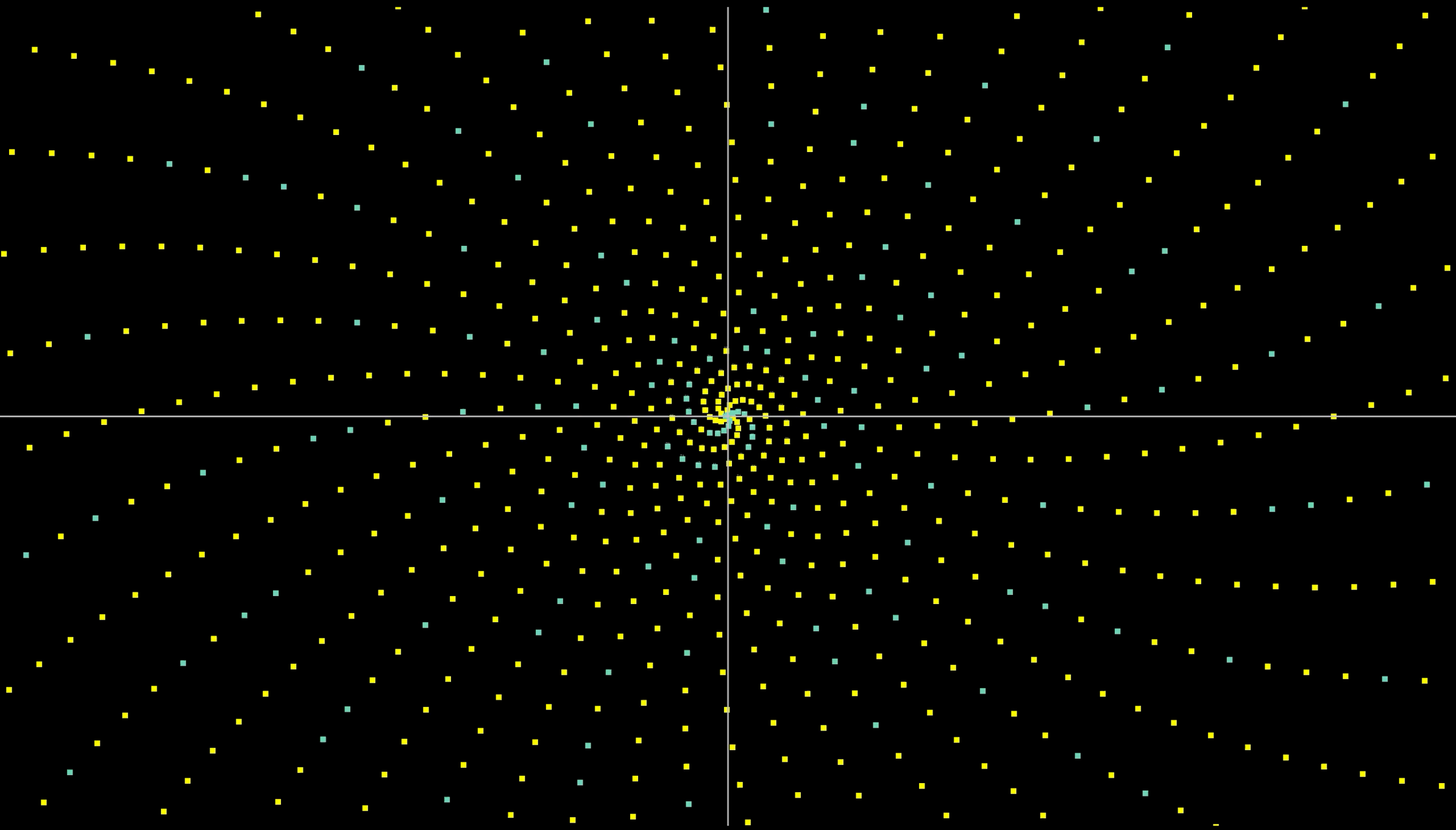
Eliminate evens

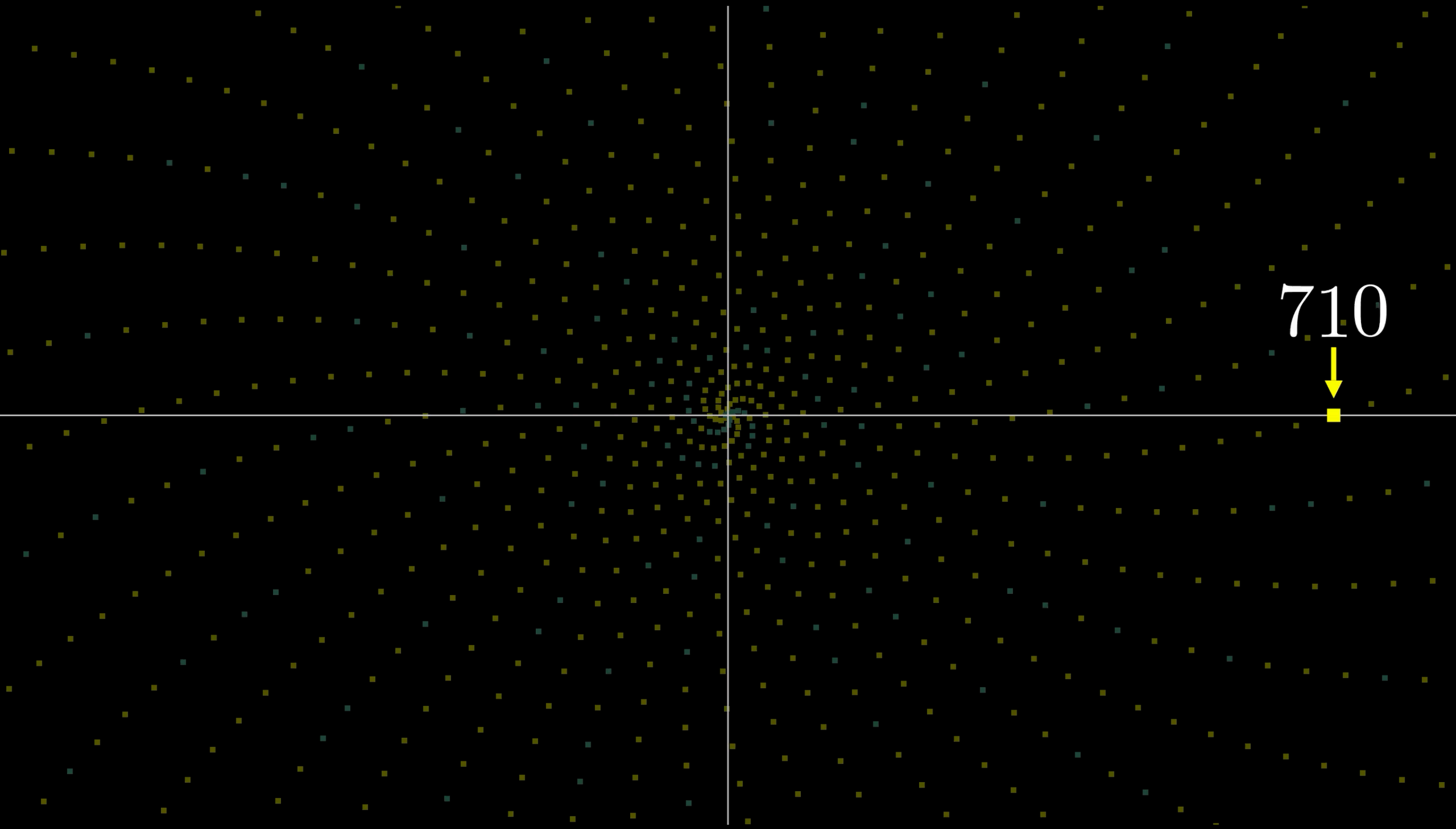


Eliminate multiples of 11

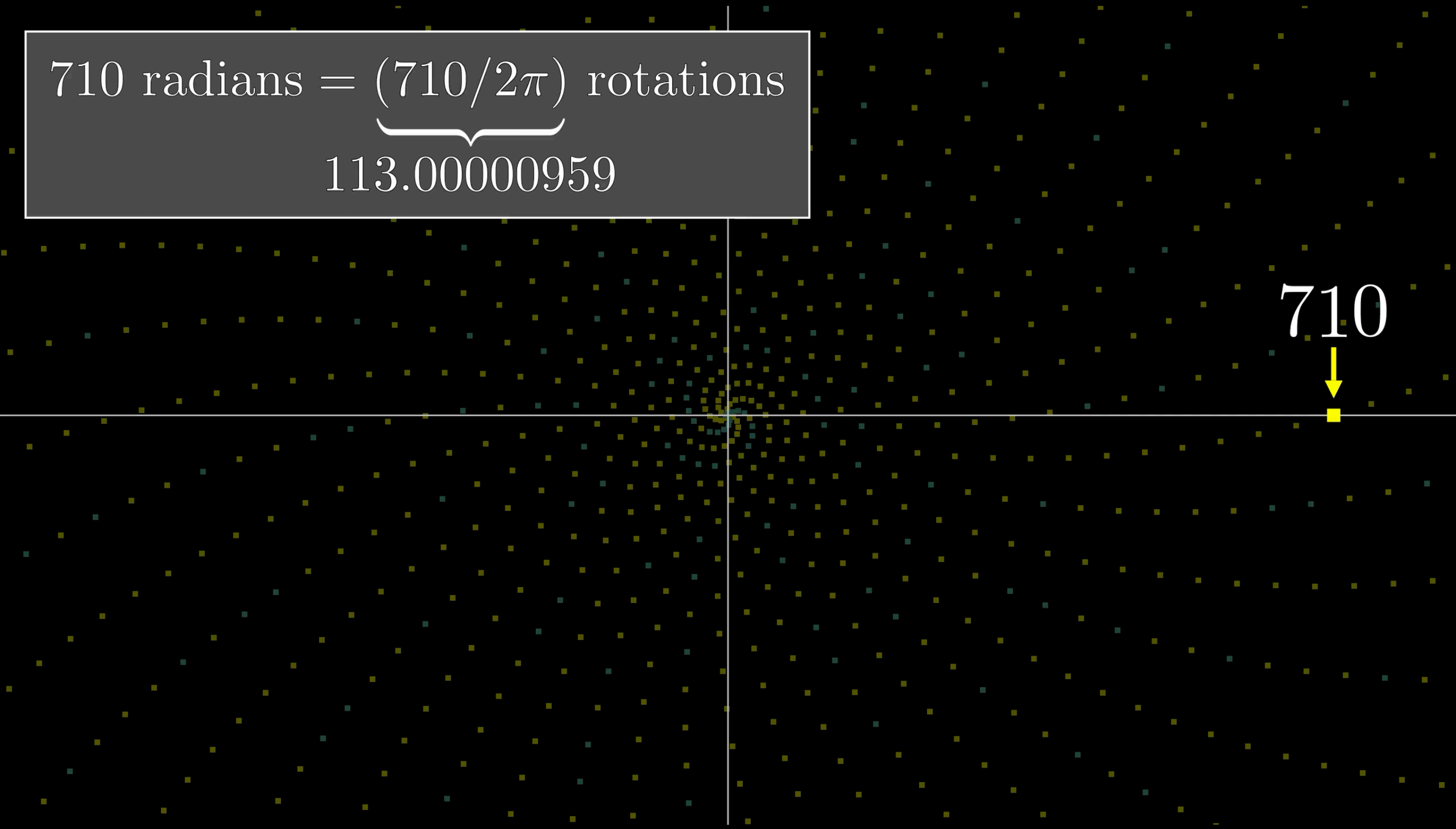







710

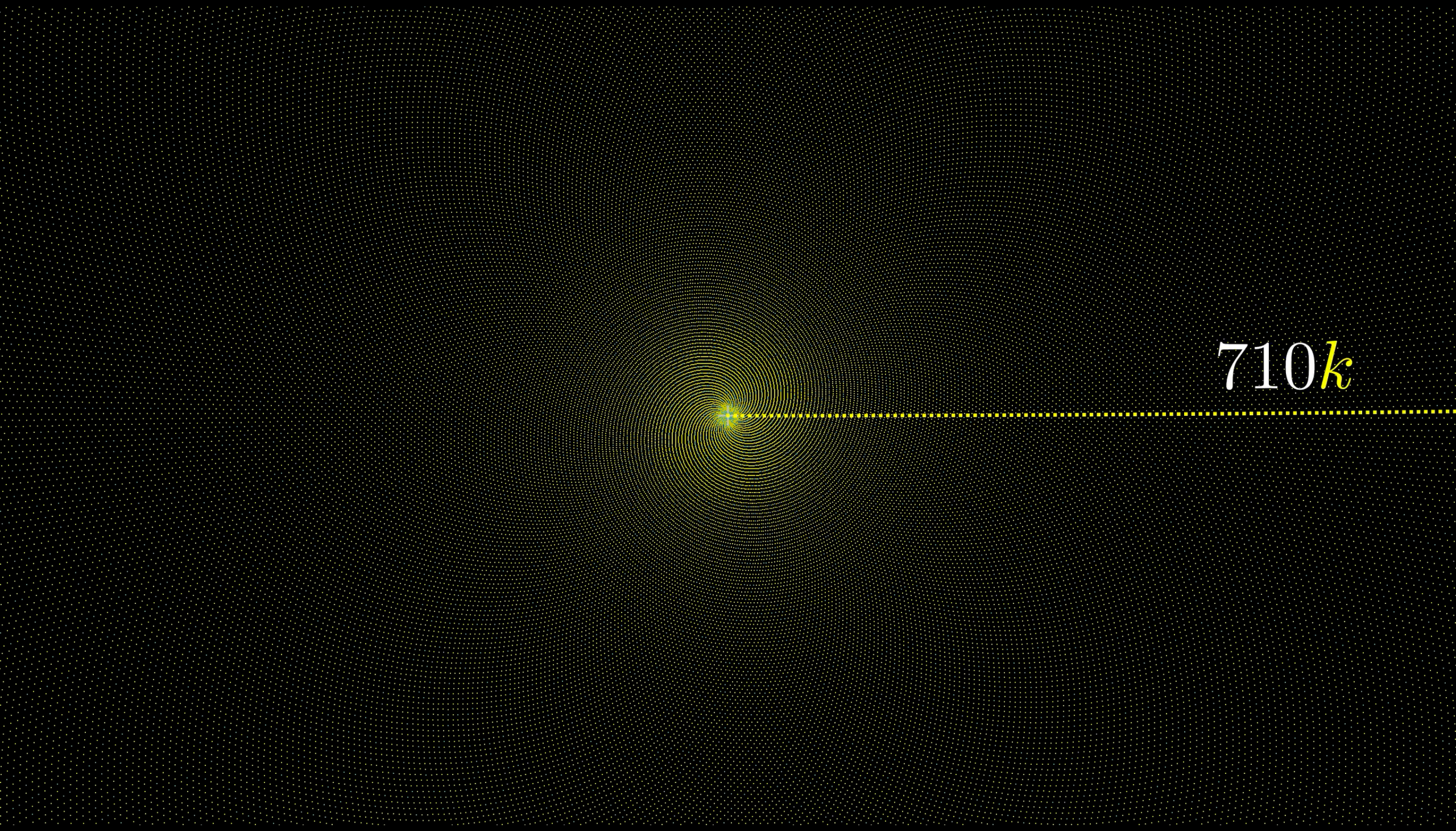
$$710 \text{ radians} = \underbrace{(710/2\pi)}_{113.00000959} \text{ rotations}$$

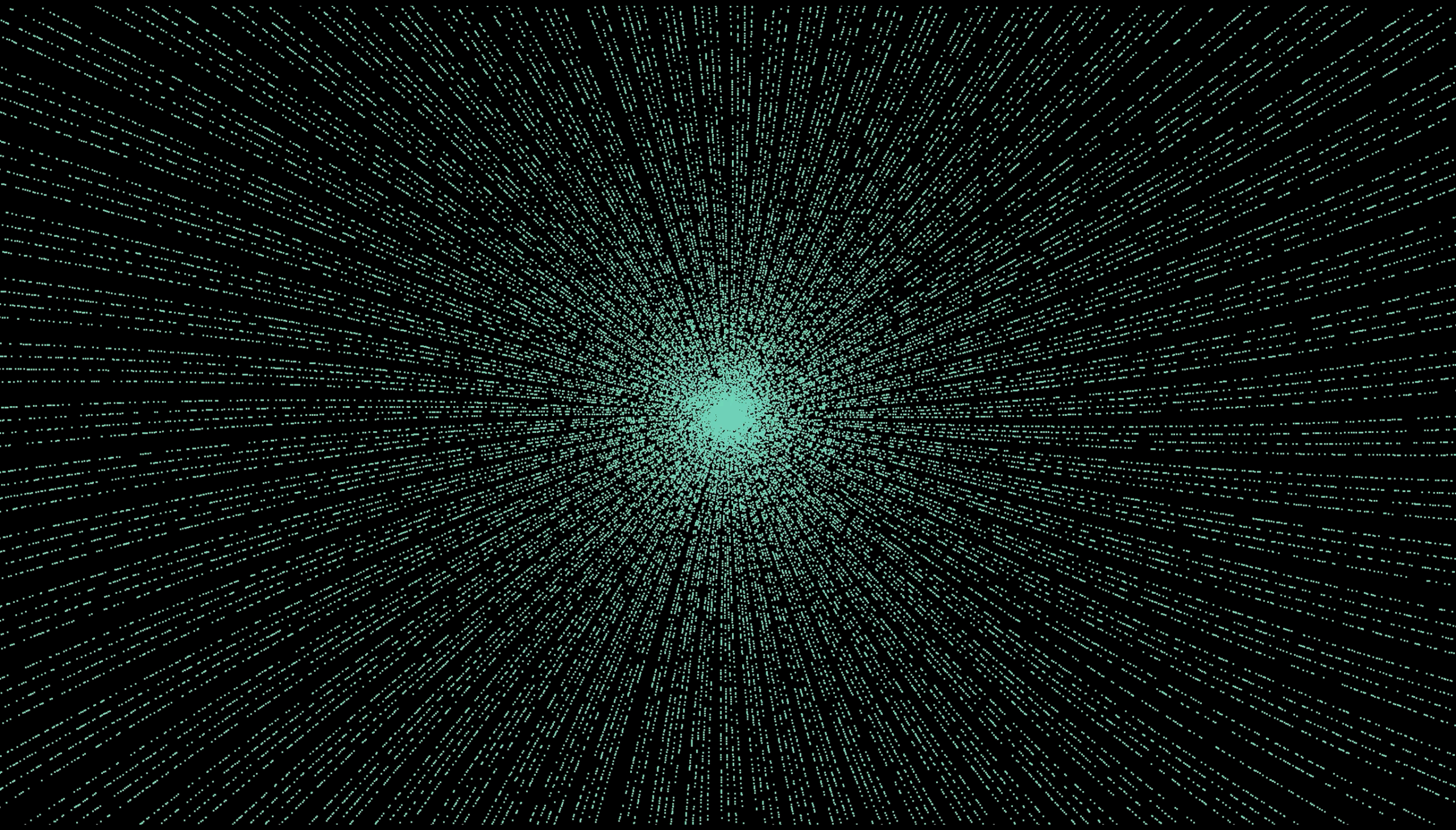


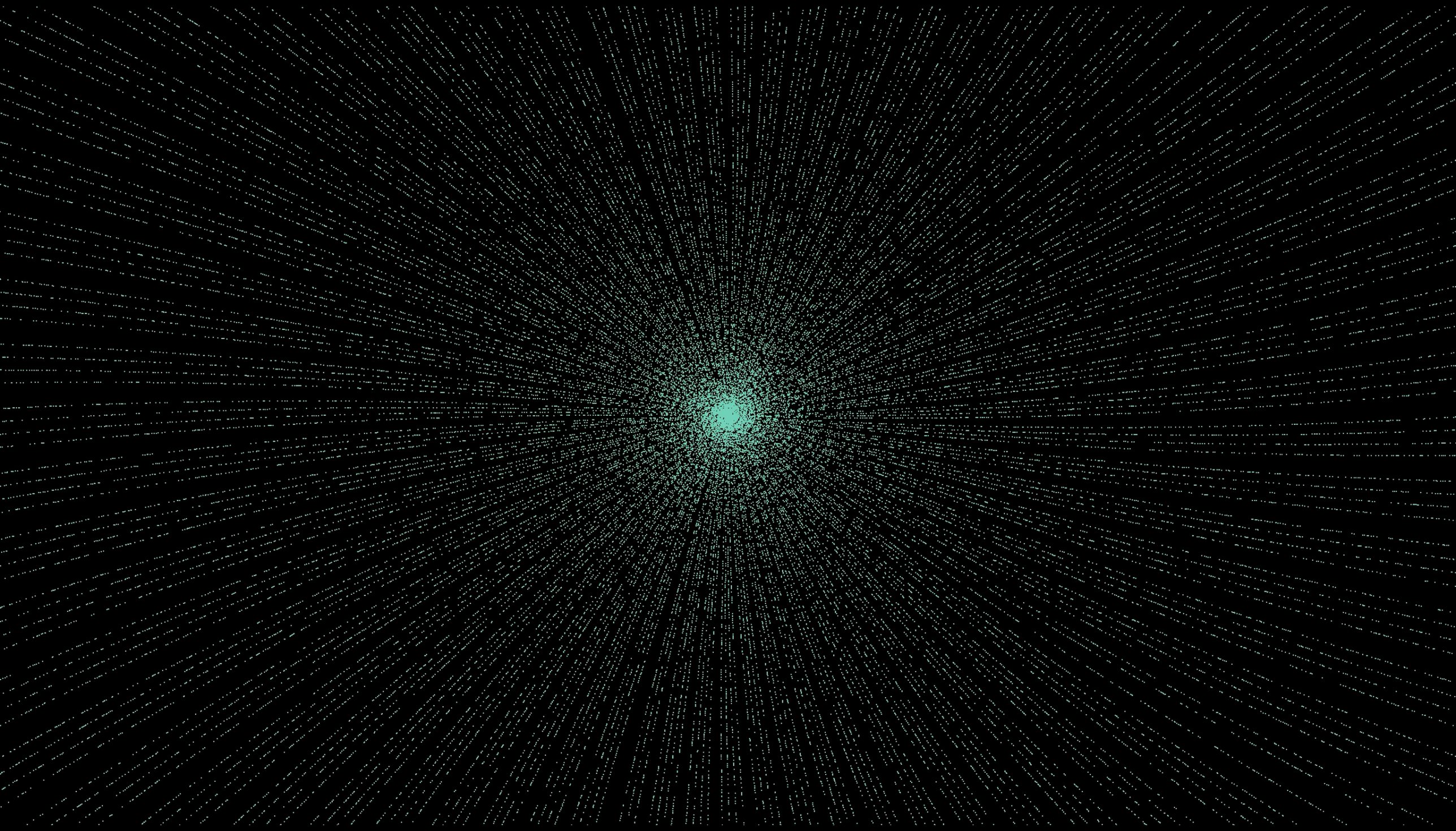
$$710 \text{ radians} = \underbrace{(710/2\pi)}_{113.00000959} \text{ rotations}$$

$$\frac{710}{113} \approx 2\pi \qquad \frac{355}{113} \approx \pi$$

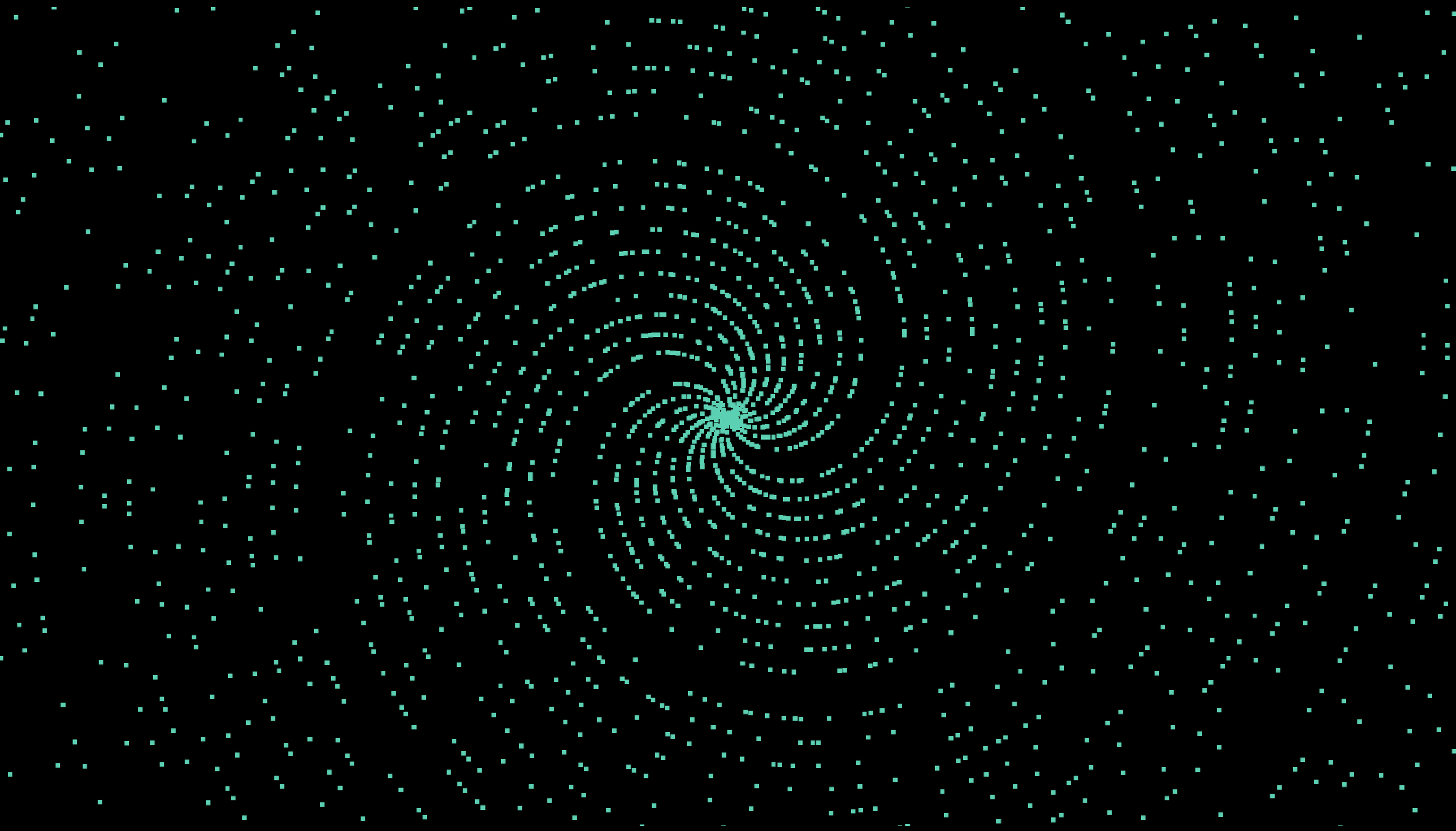
710


710k

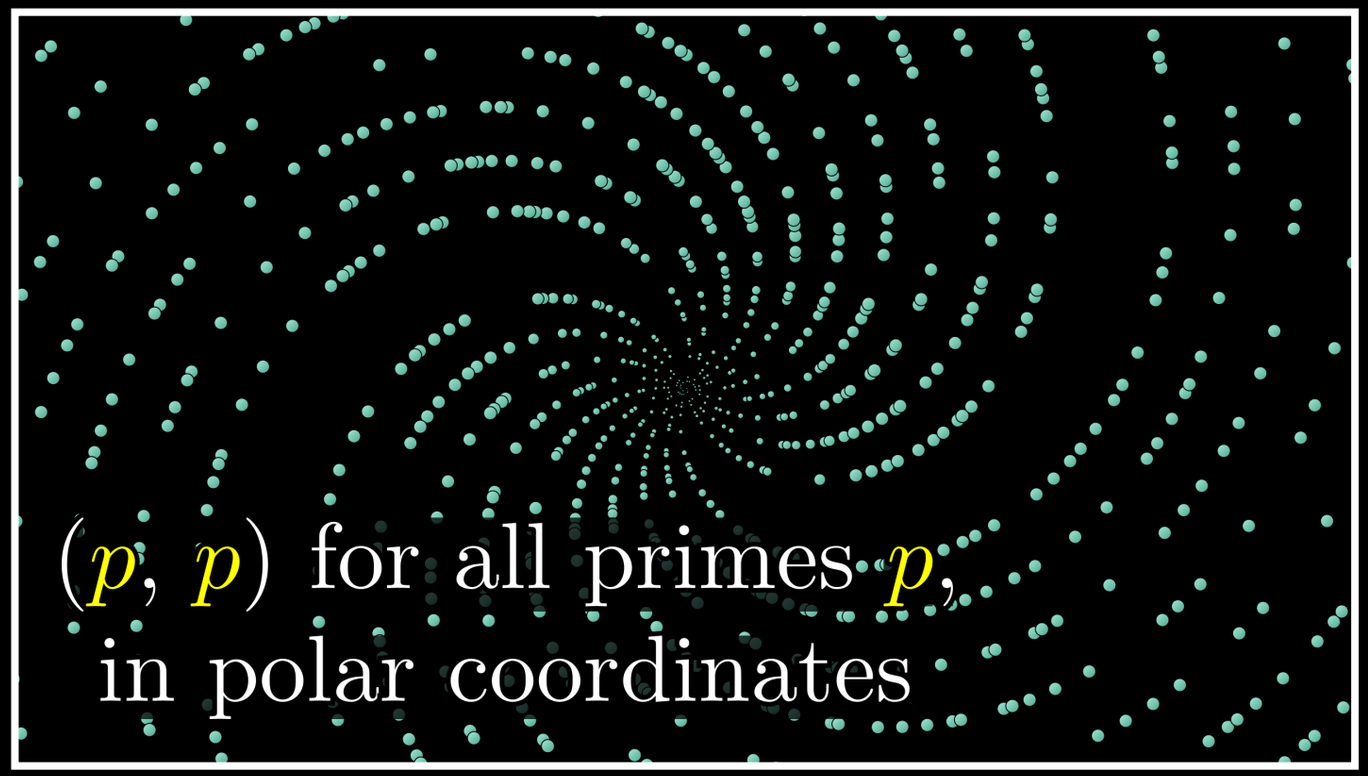




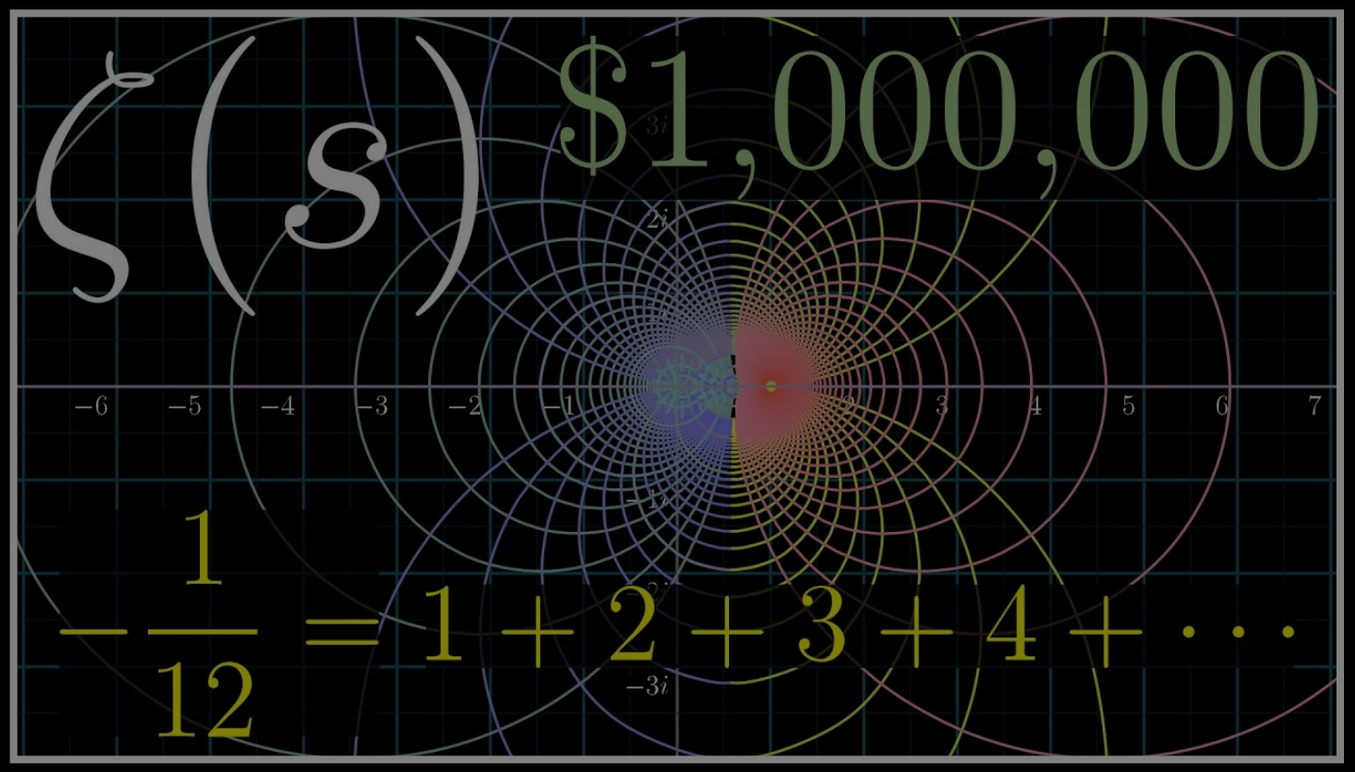
Evenly distributed?



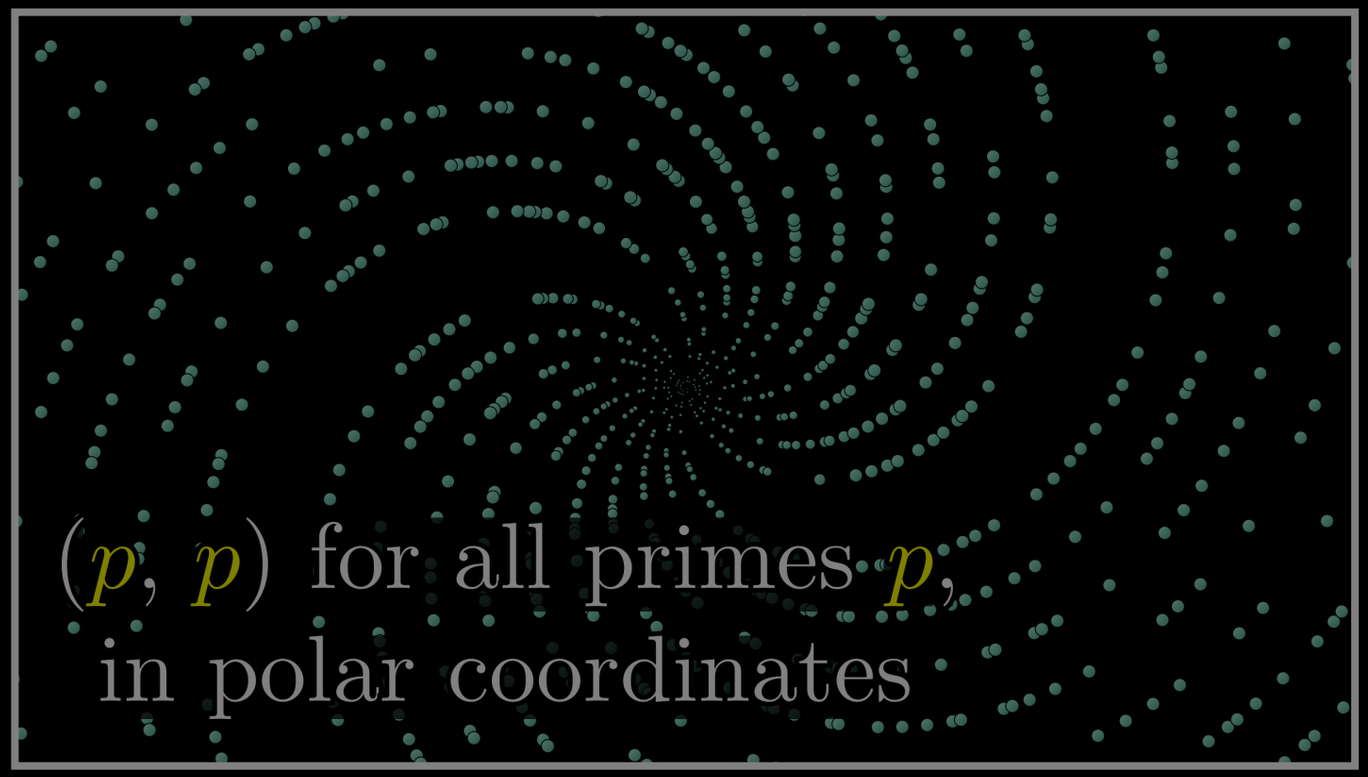
Simpler visual for the distribution of primes mod 44



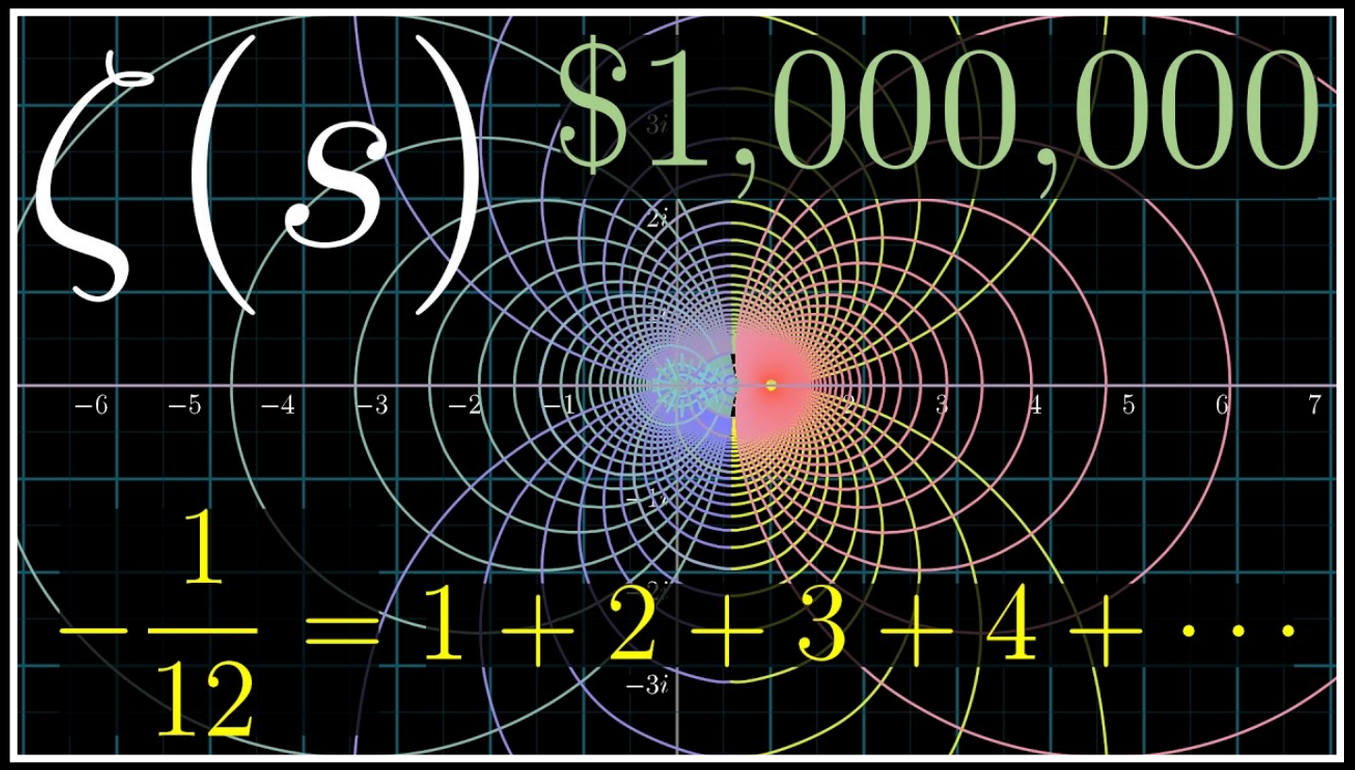
Use visuals to inspire



Use visuals to elucidate

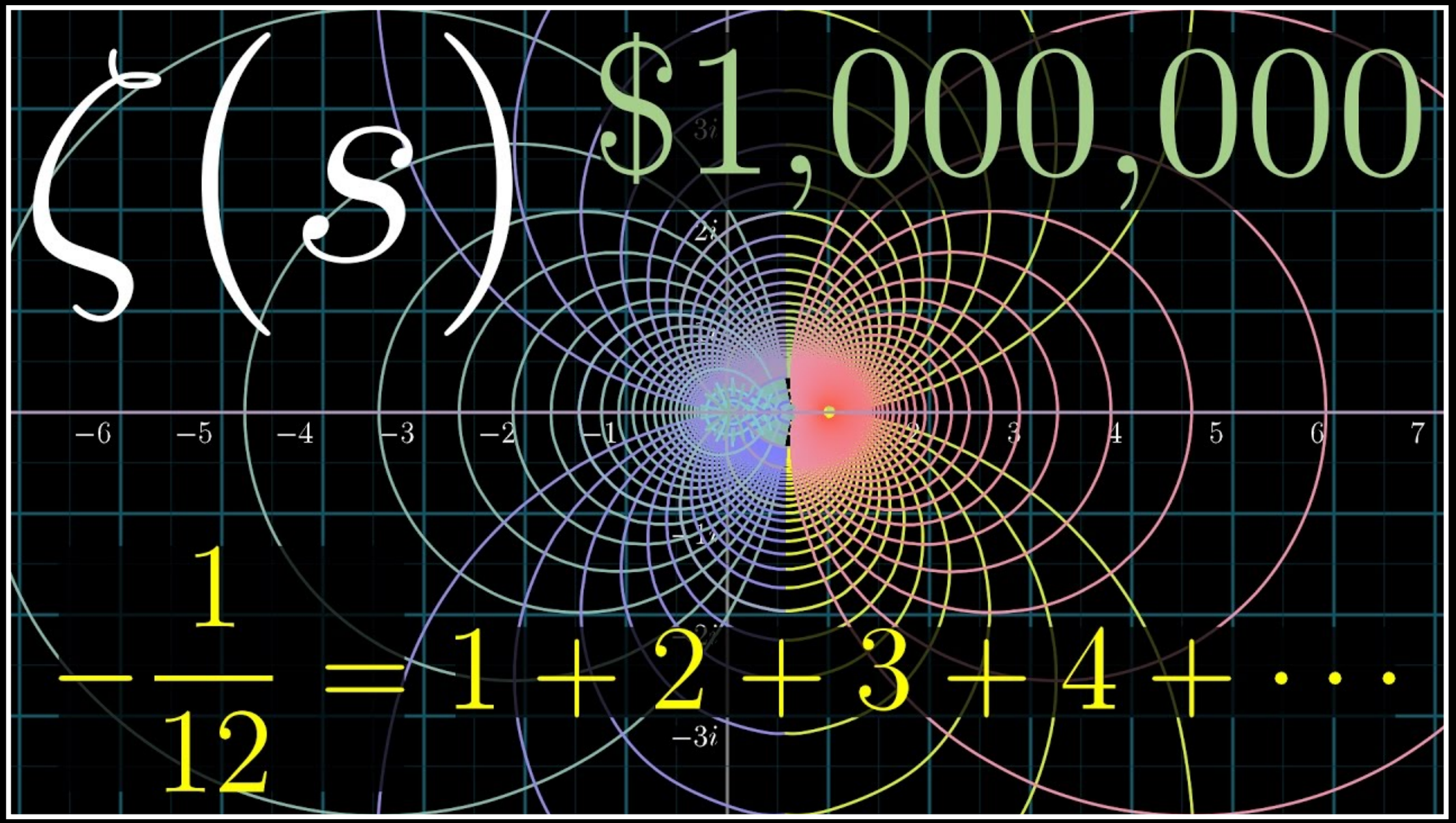


Use visuals to inspire



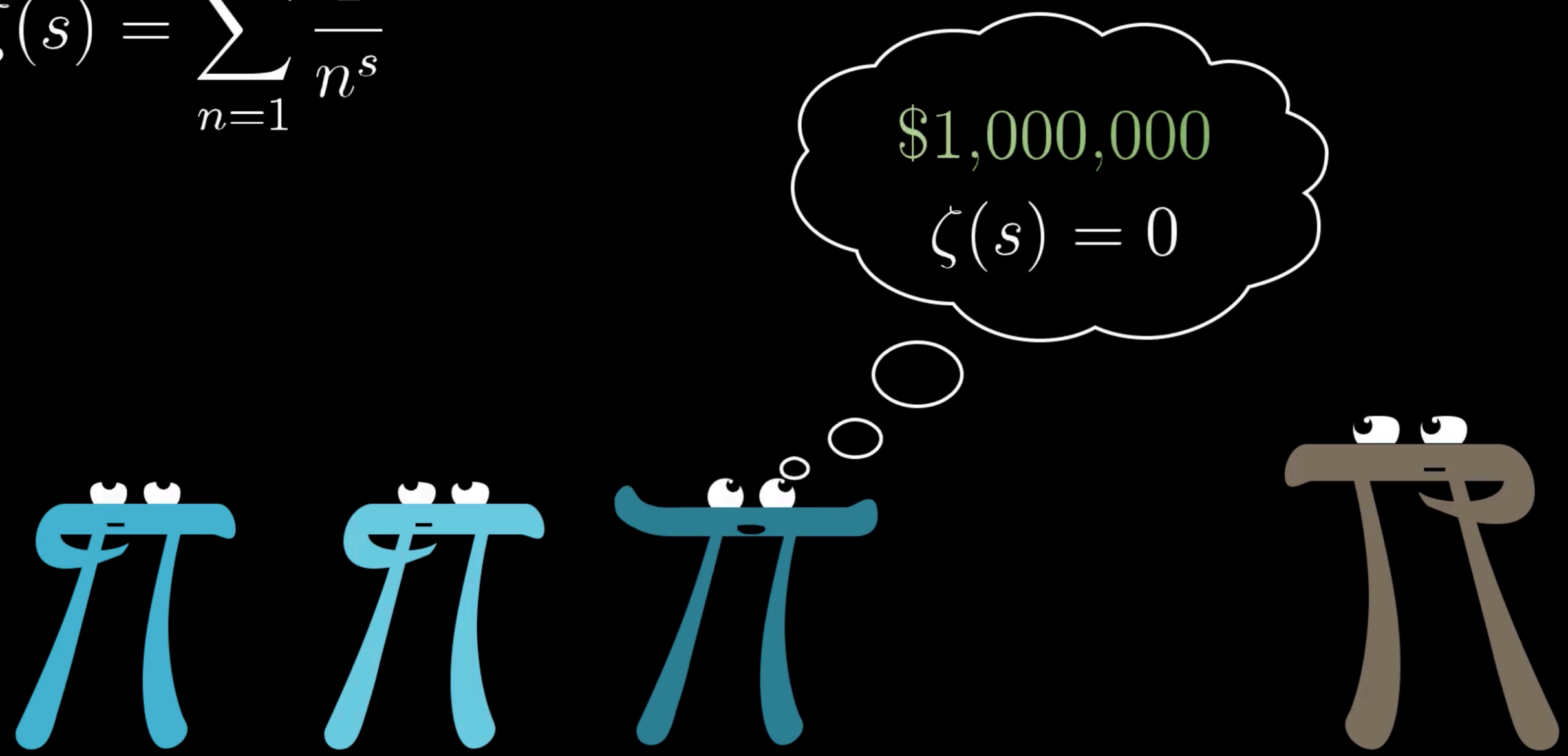
Use visuals to elucidate

Video from 2016



Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

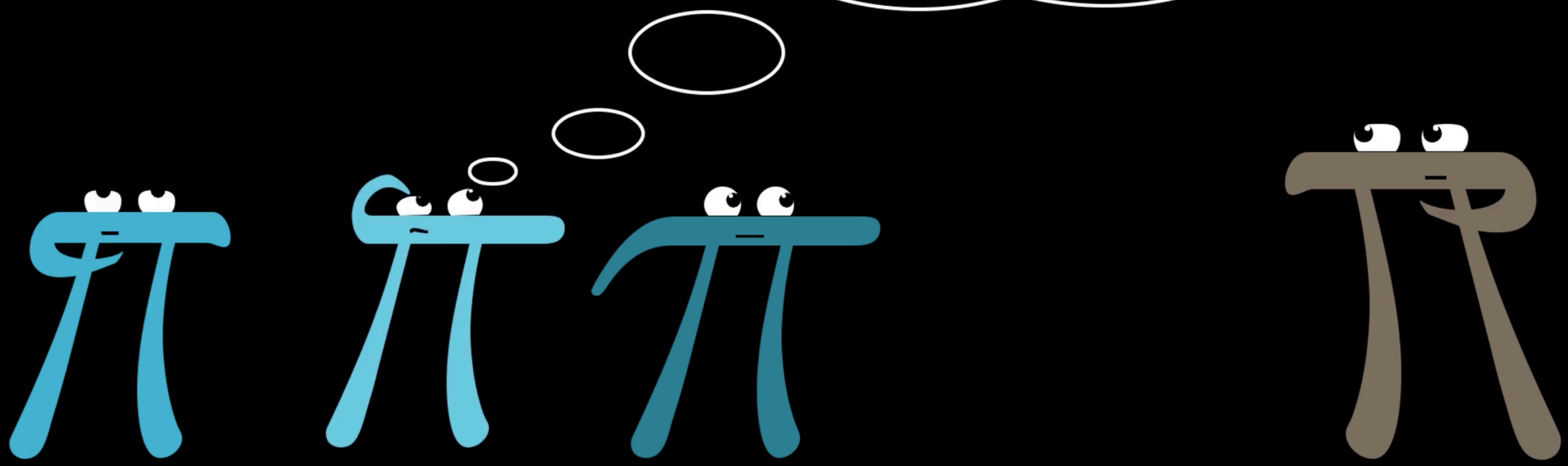


Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

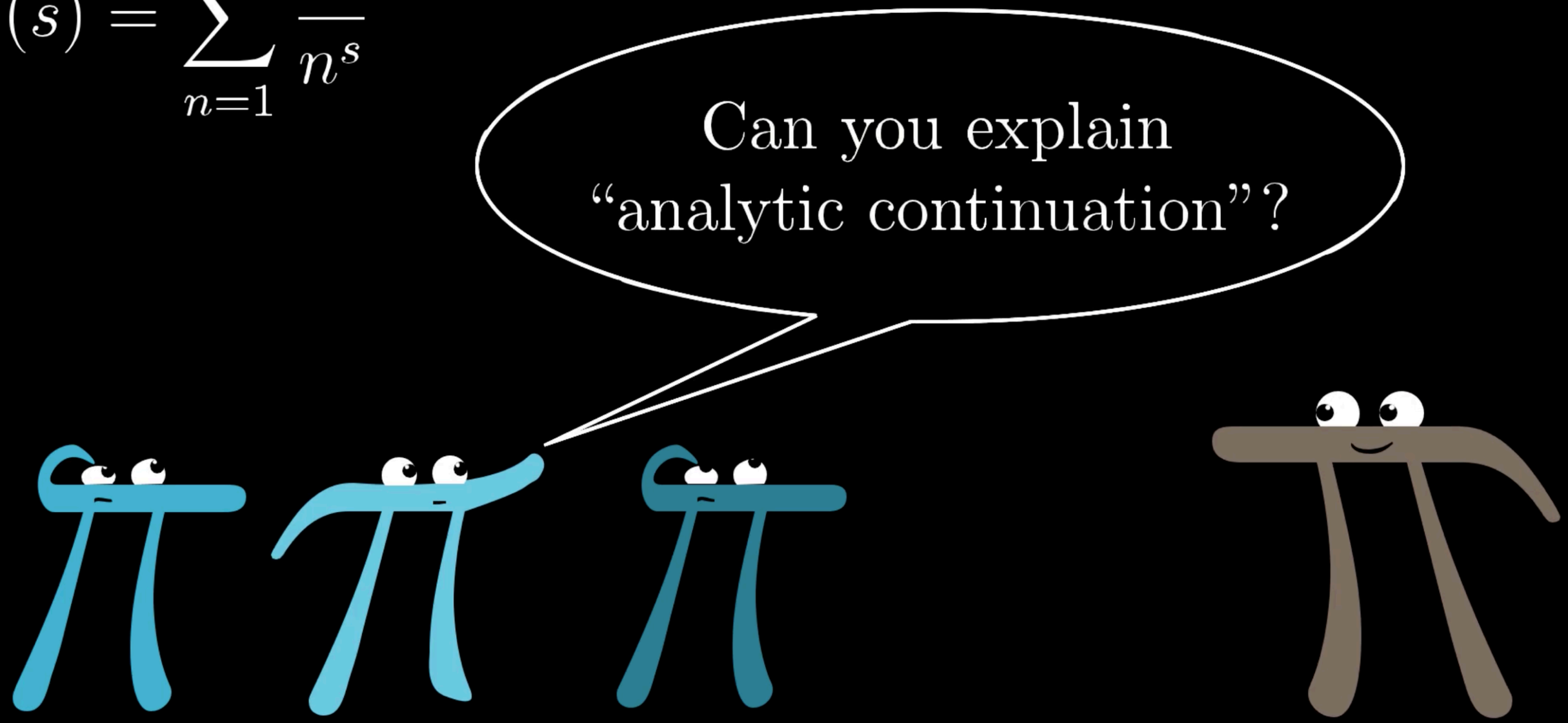
$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$$

$$\zeta(-1) = -\frac{1}{12}$$

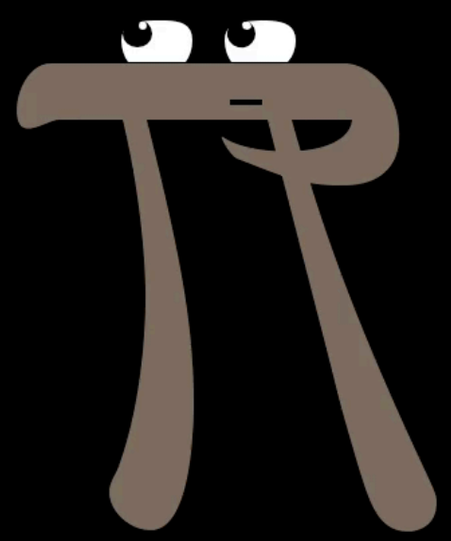


Riemann zeta function

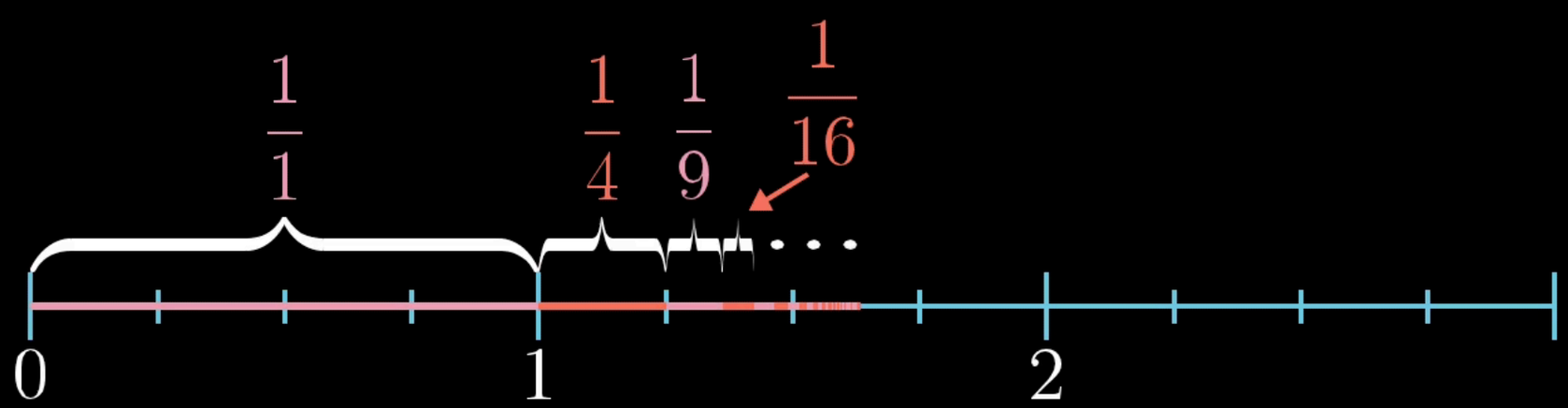
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

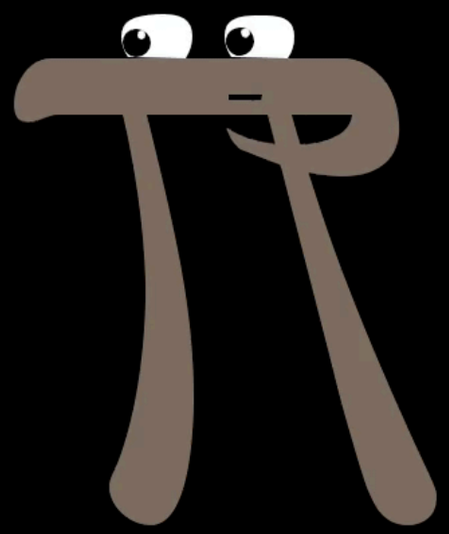


$$\zeta(s) = \overbrace{\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots}^{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}}$$

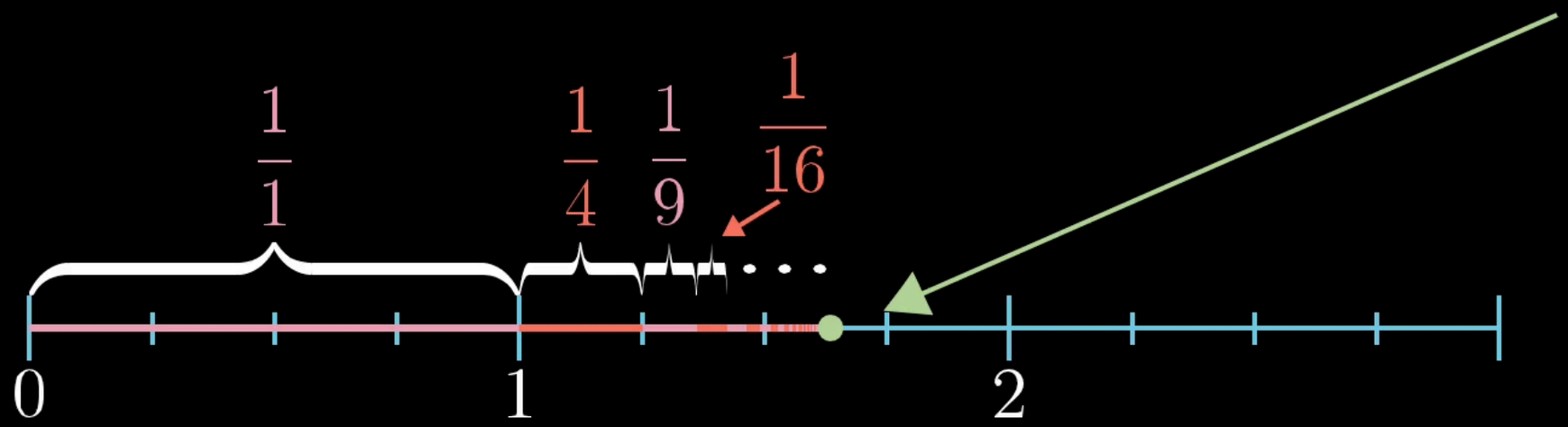


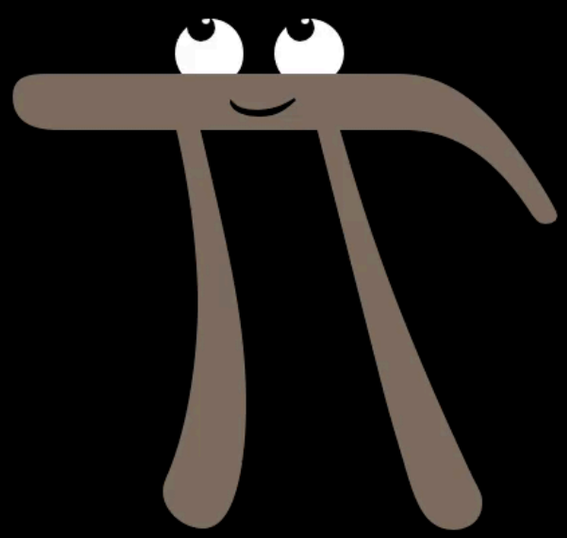
$$\zeta(2) = \overbrace{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots}^{\sum_{n=1}^{\infty} \frac{1}{n^2}}$$





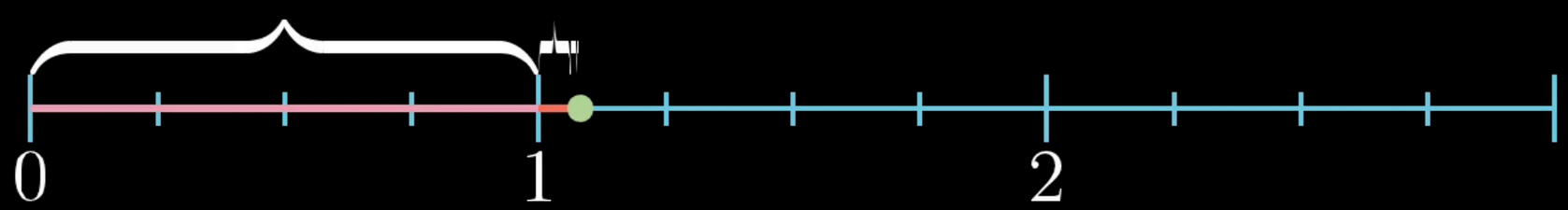
$$\zeta(2) = \overbrace{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots}^{\sum_{n=1}^{\infty} \frac{1}{n^2}} = \frac{\pi^2}{6} \approx 1.645$$

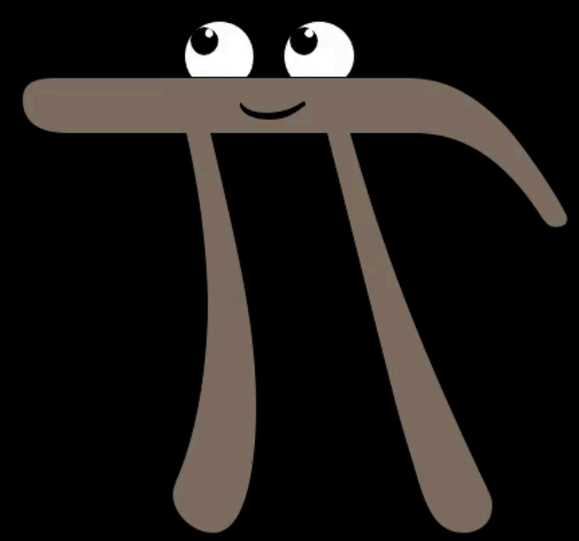




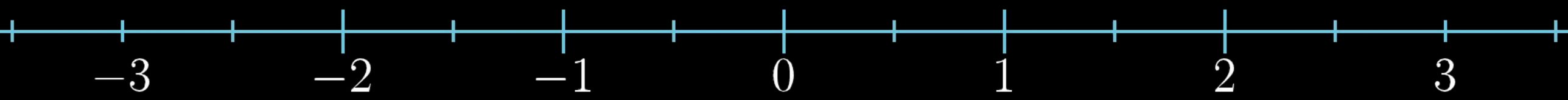
$$\zeta(4) = \overbrace{\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots}^{\sum_{n=1}^{\infty} \frac{1}{n^4}} = \frac{\pi^4}{90}$$

Getting smaller

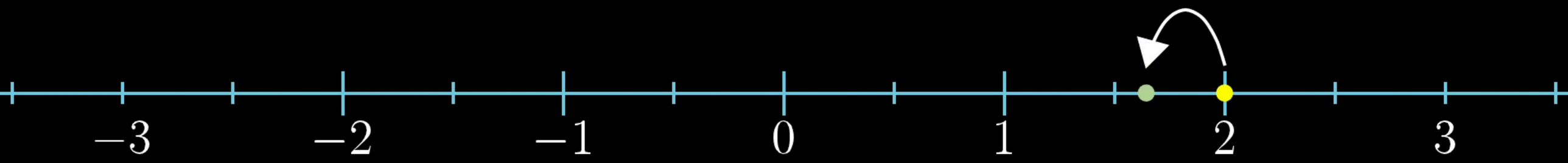




$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

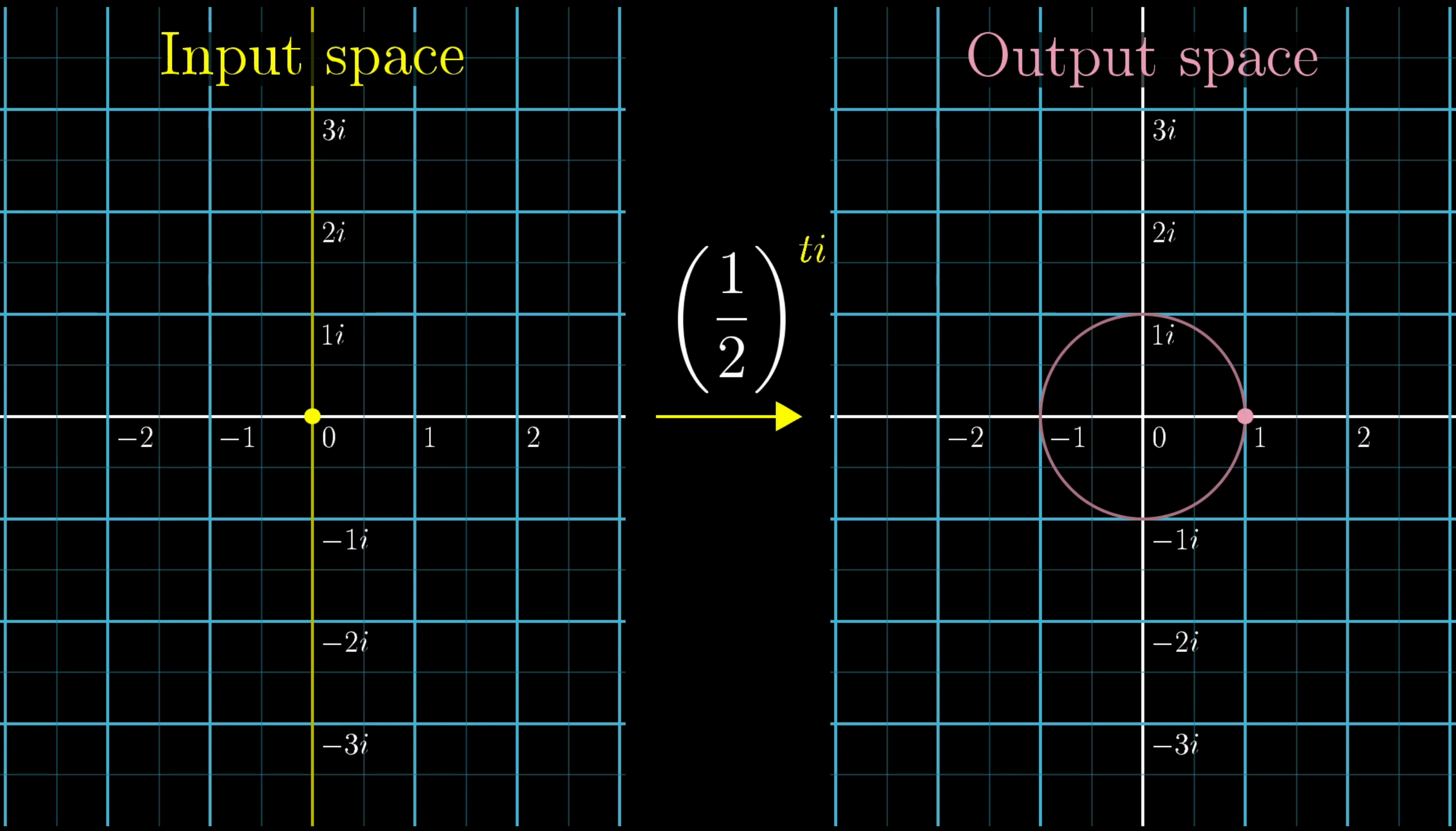


$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

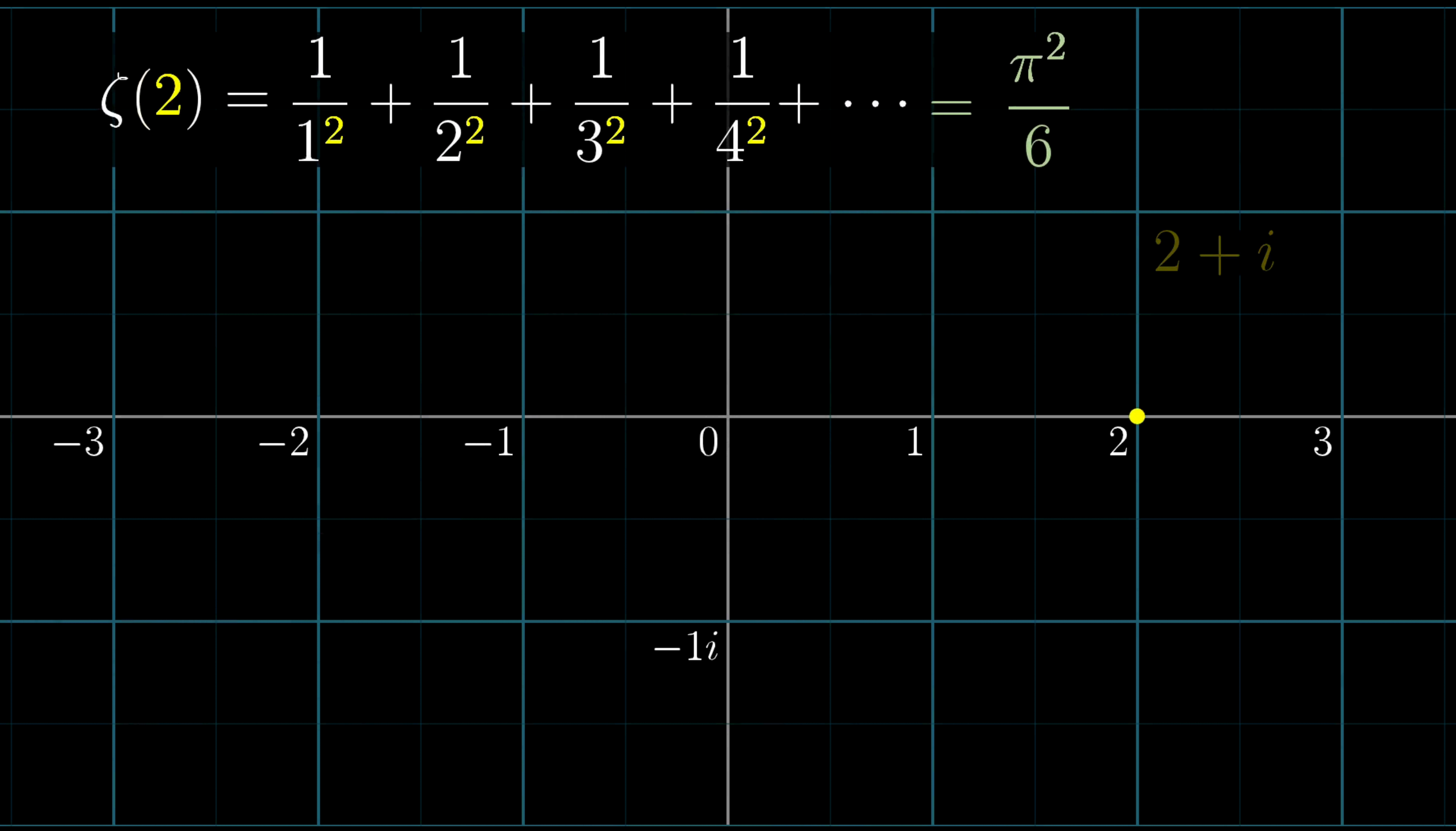


$$\left(\frac{1}{2}\right)^{2+i}$$

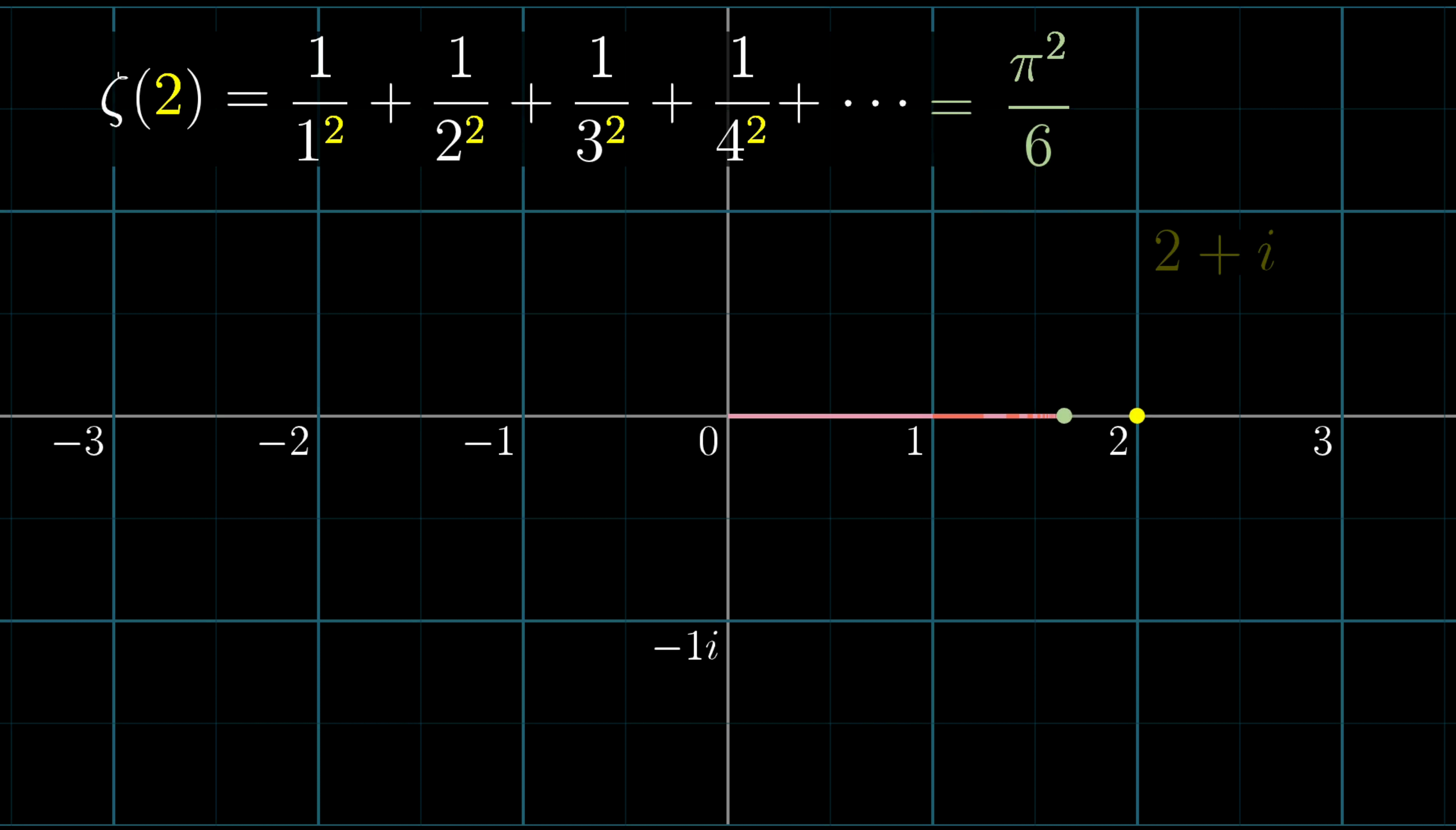
$$\underbrace{\left(\frac{1}{2}\right)^2}_{\text{We understand this}} \left(\frac{1}{2}\right)^i$$


Input space
Output space
$\left(\frac{1}{2}\right)^{ti}$
$3i$
$2i$
$1i$
$-1i$
$-2i$
$-3i$
-2
-1
0
1
2

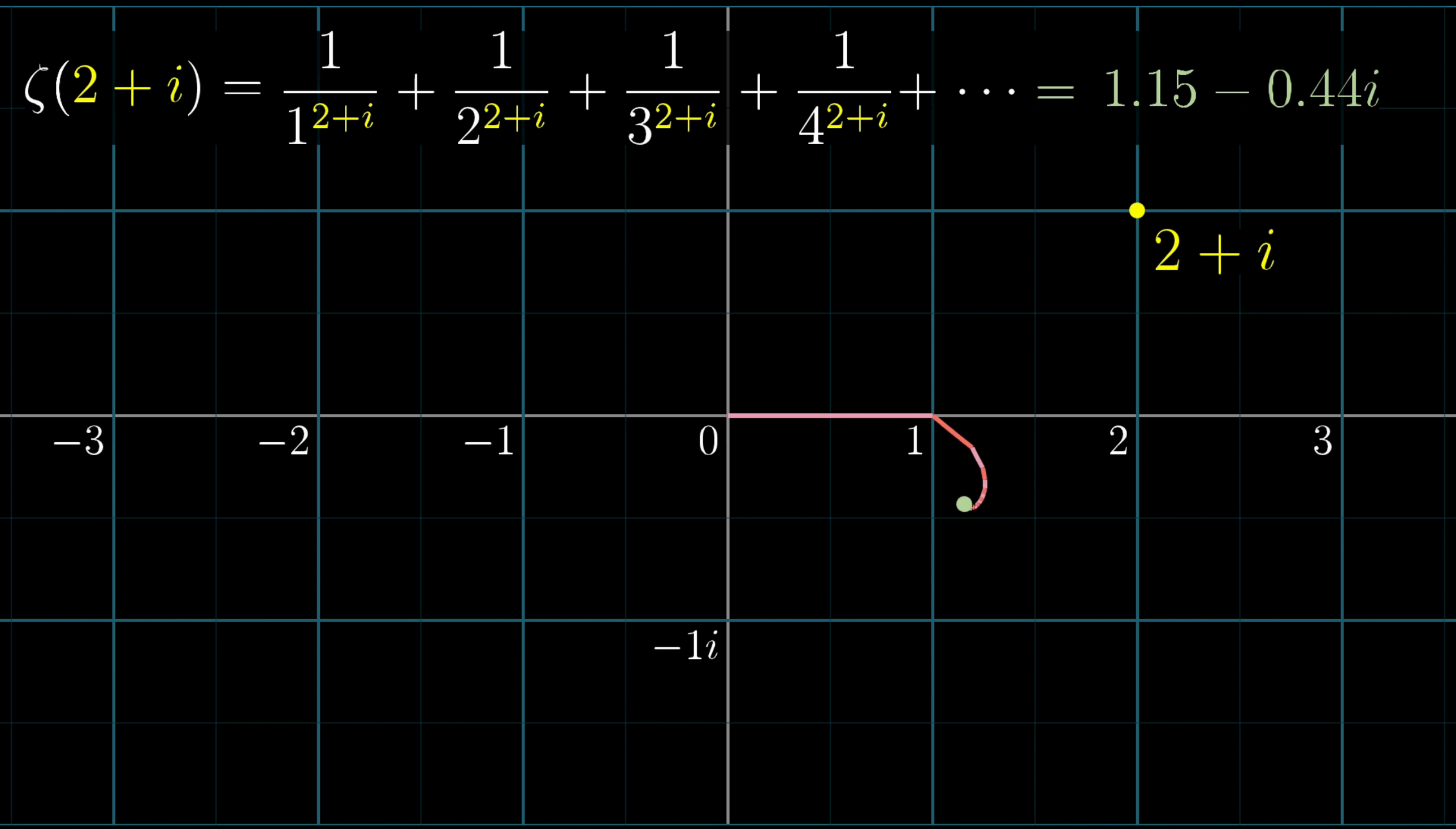
$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$



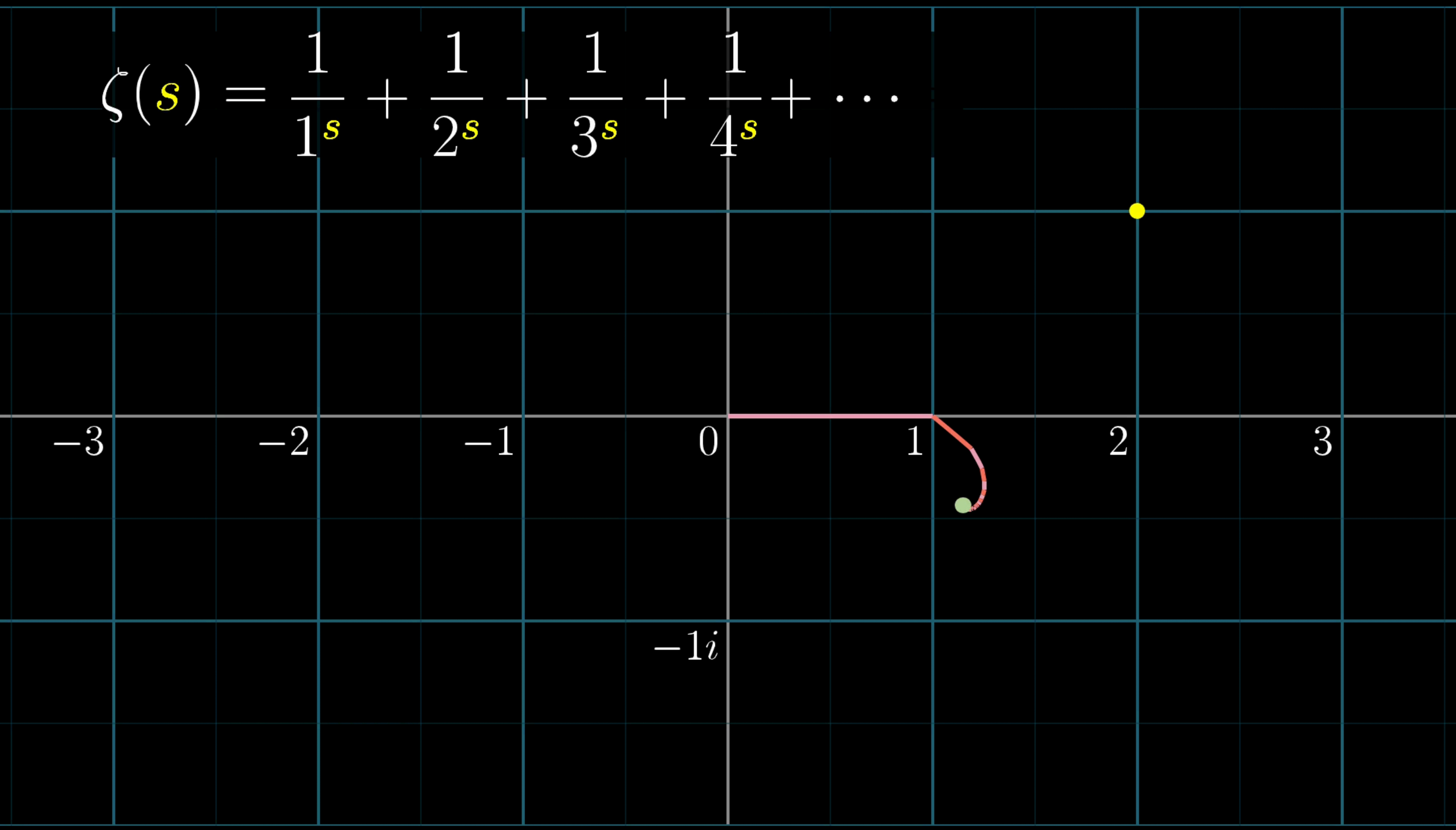
$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$



$$\zeta(2+i) = \frac{1}{1^{2+i}} + \frac{1}{2^{2+i}} + \frac{1}{3^{2+i}} + \frac{1}{4^{2+i}} + \cdots = 1.15 - 0.44i$$

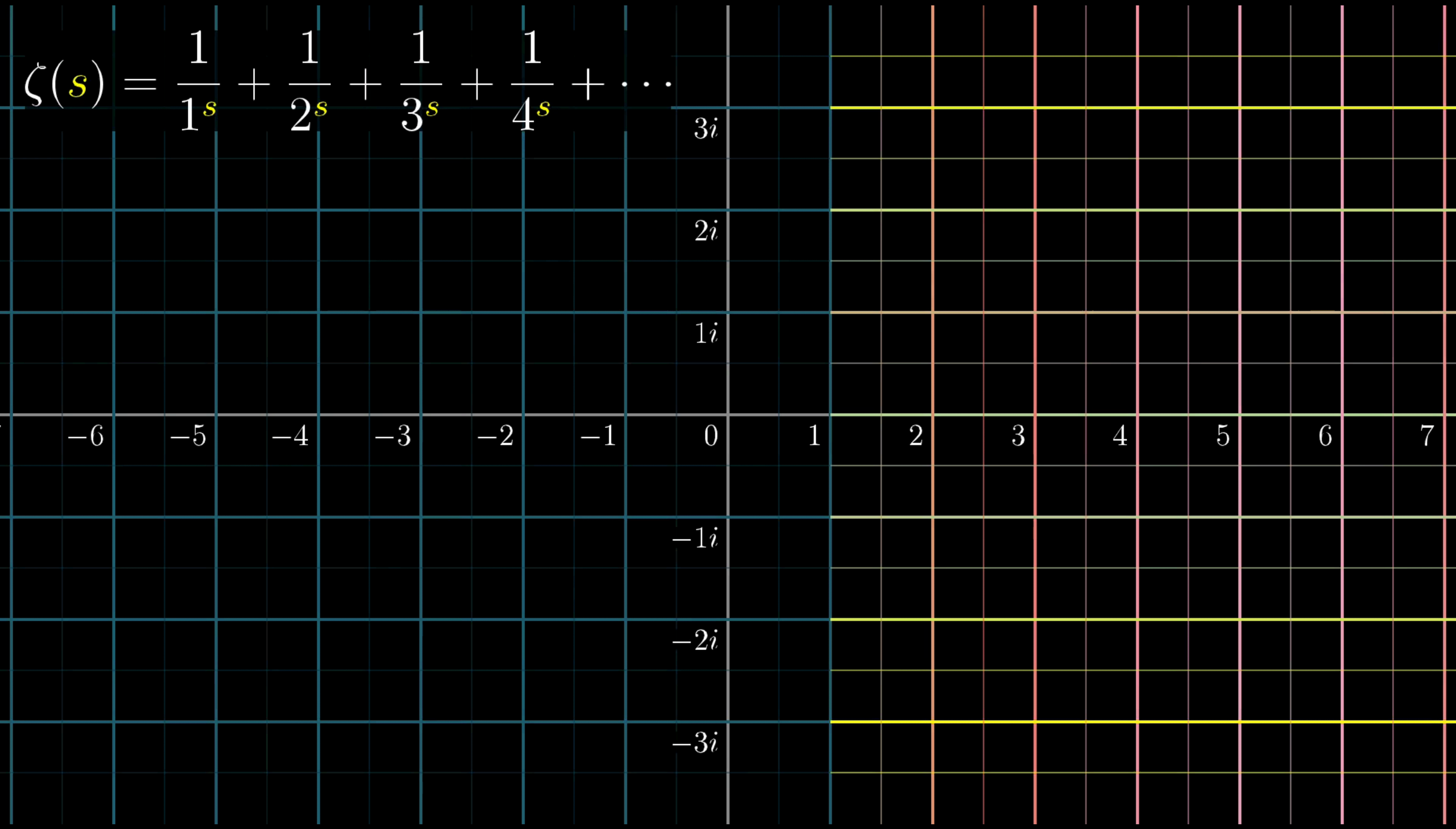


$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

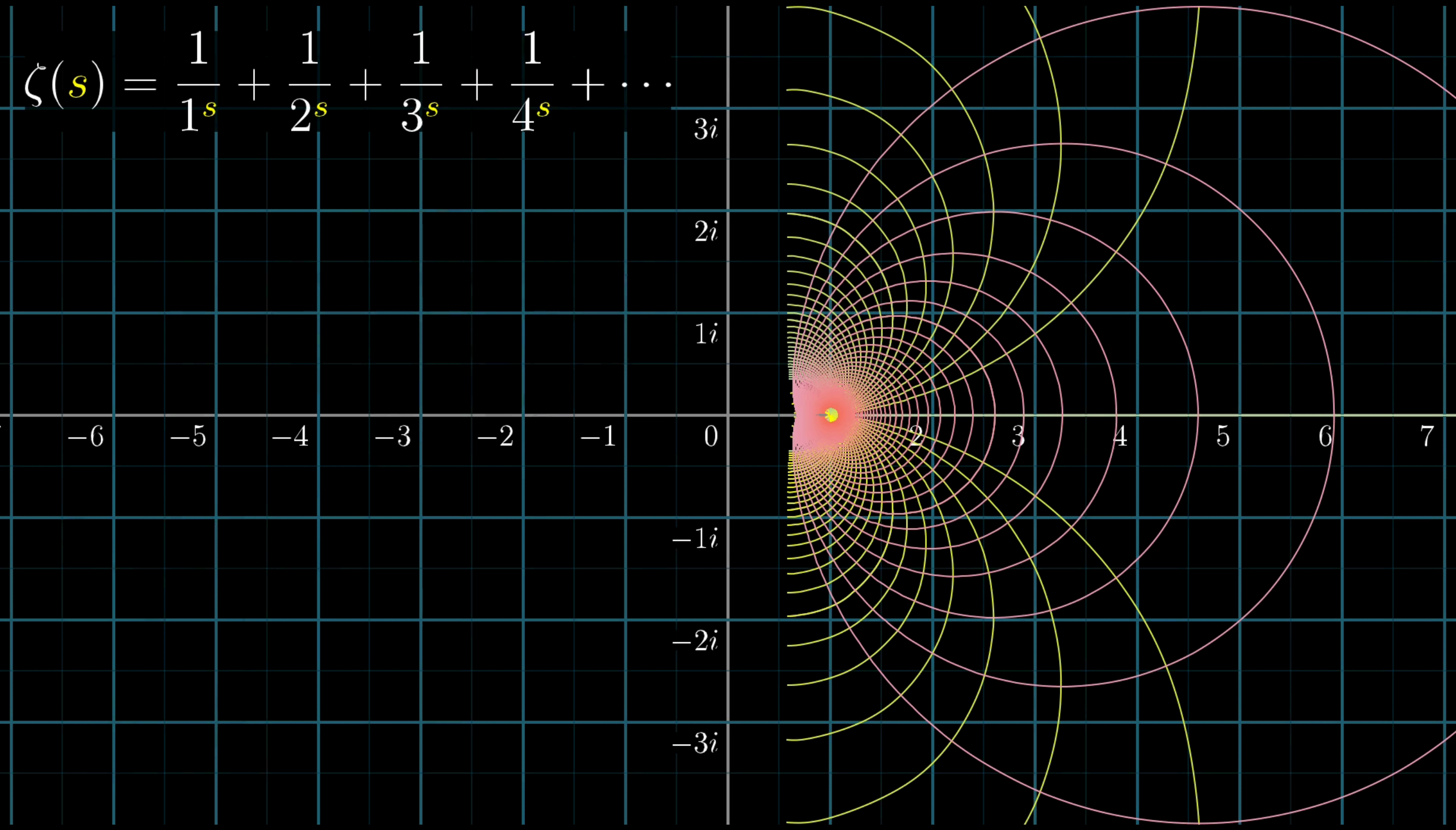


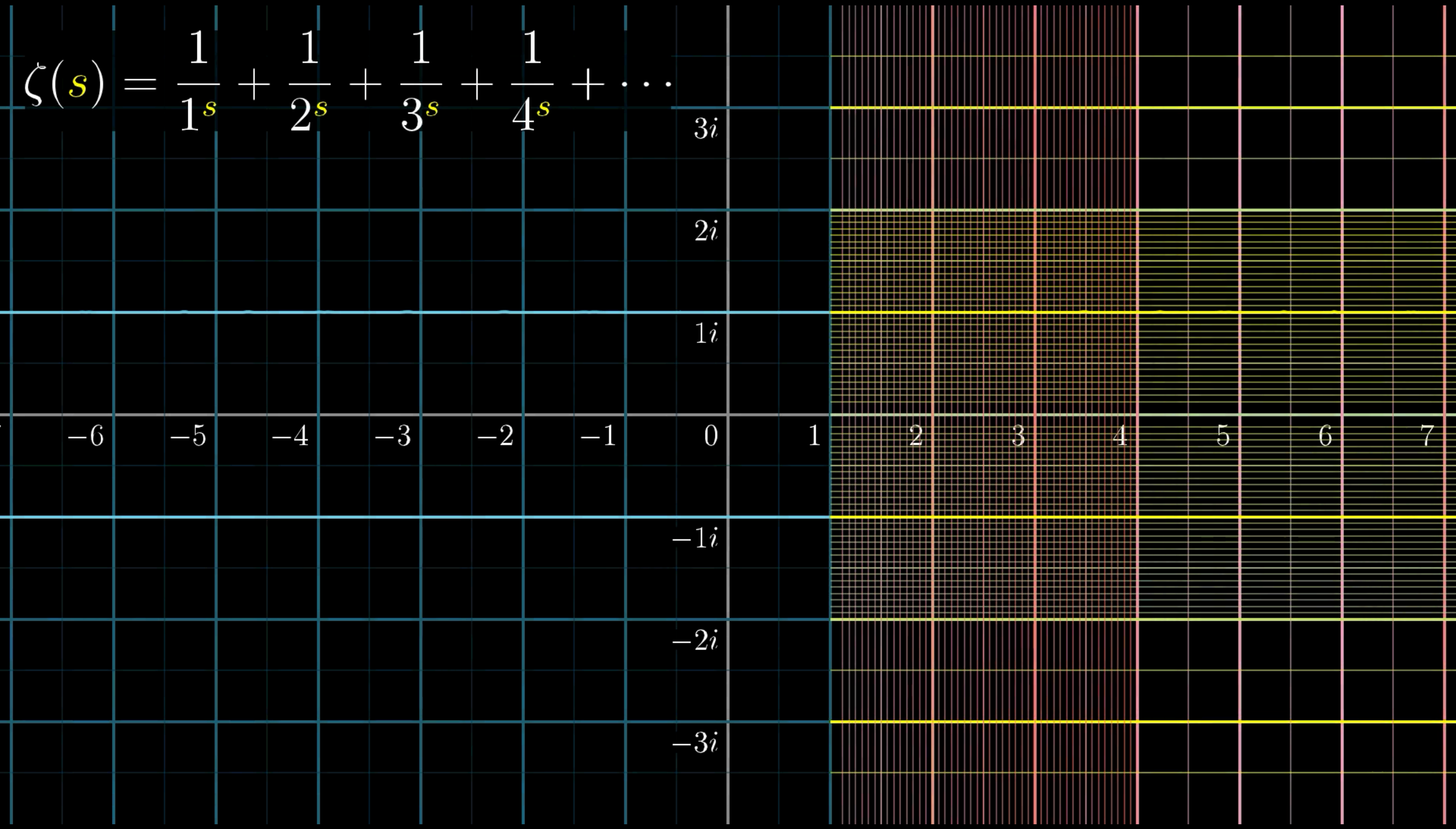
$$\zeta(s) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} & \text{if } \mathrm{Re}(s) > 1 \\ \underbrace{\text{Something else...}}_{\text{What to put here?}} & \text{if } \mathrm{Re}(s) \leq 1 \end{cases}$$

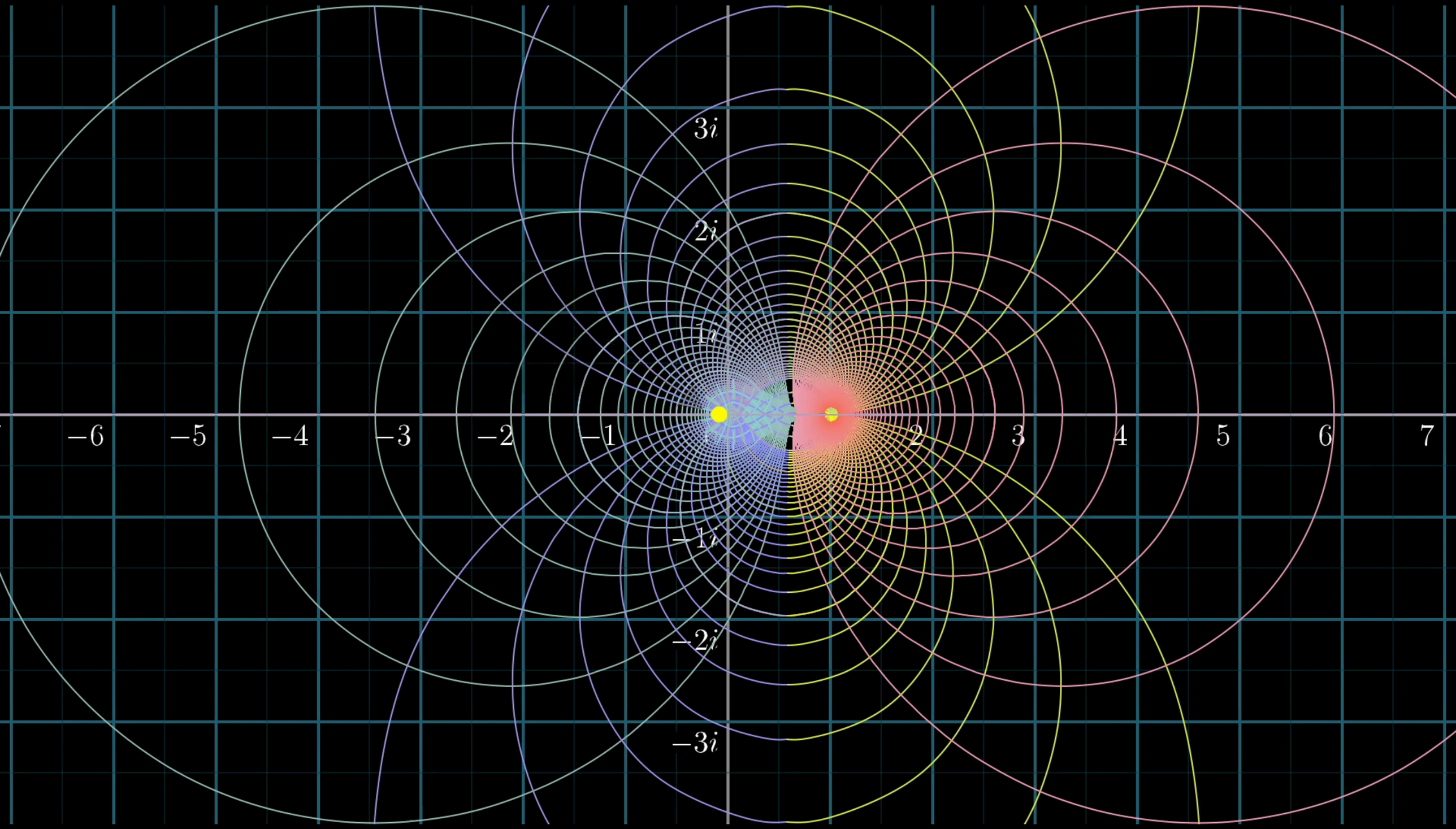
$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

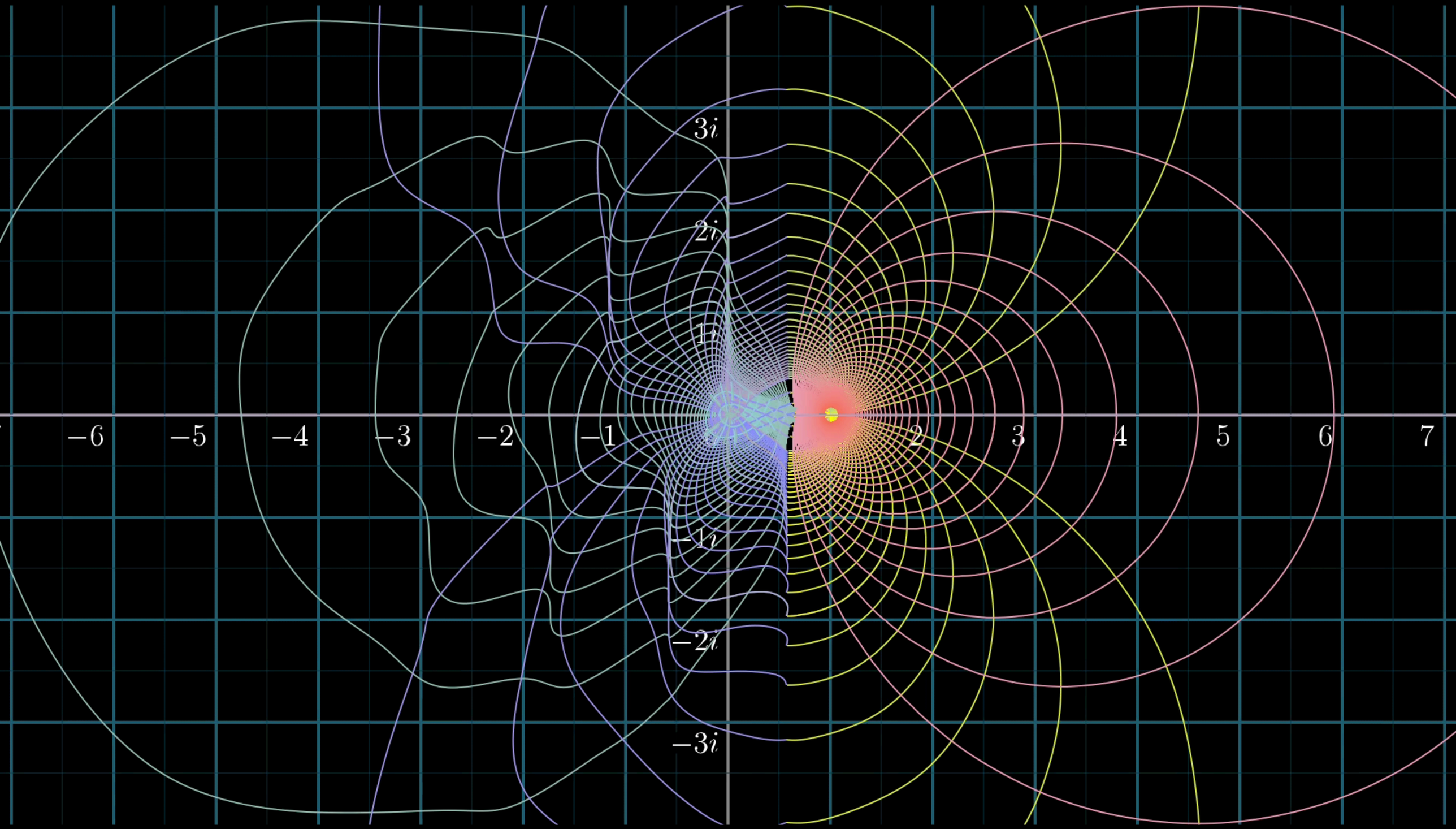


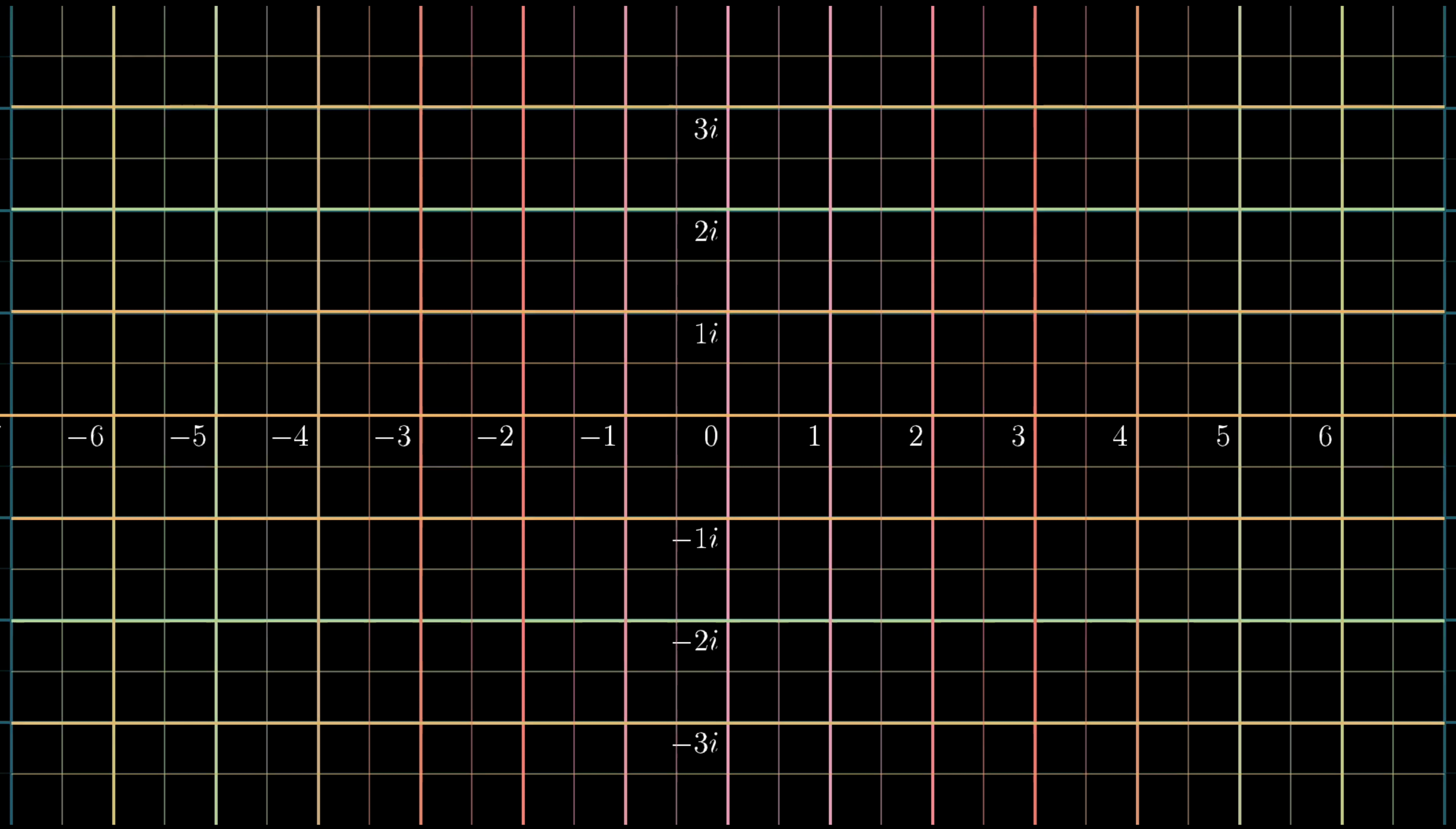
$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$



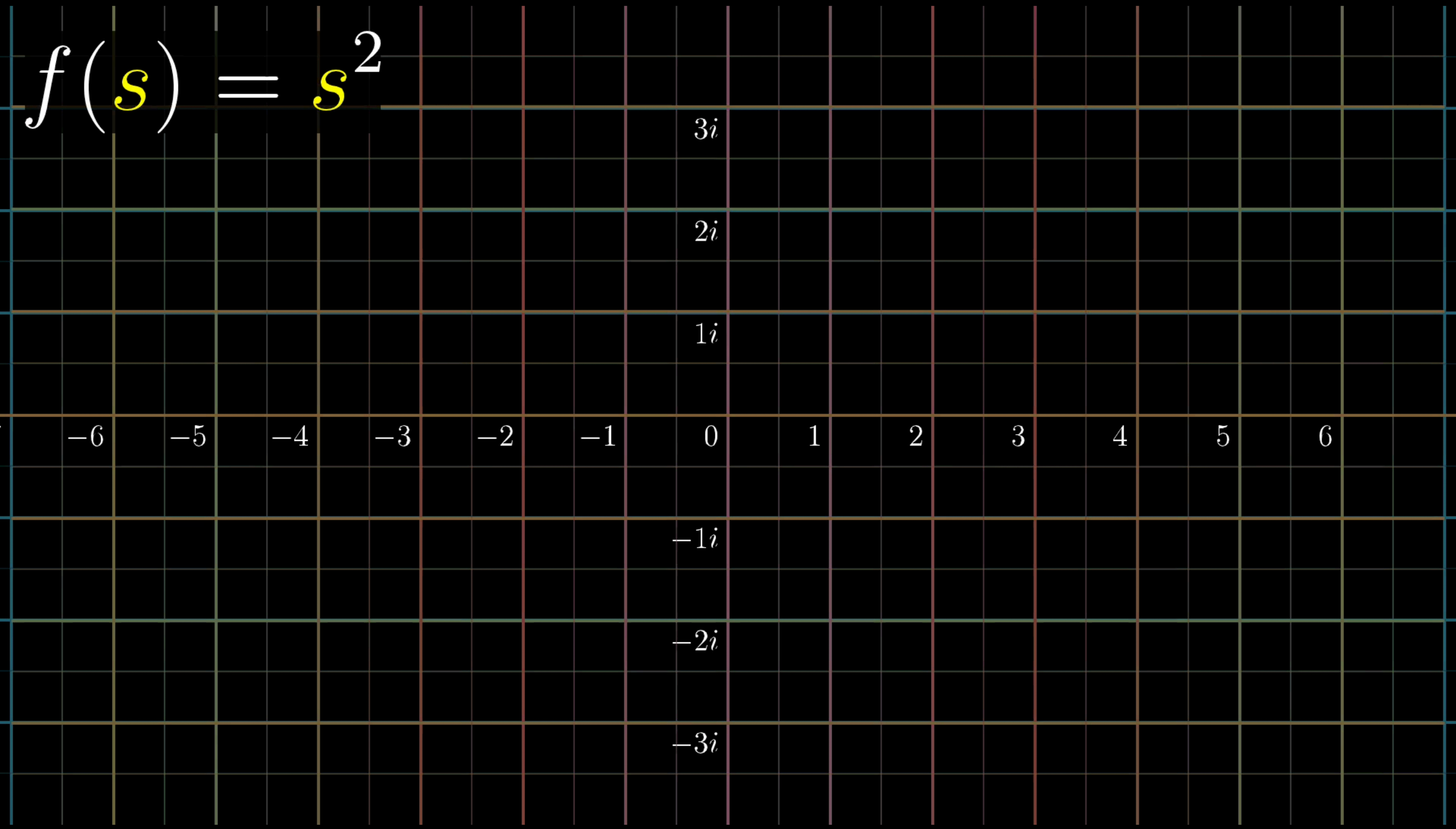

$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$
$3i$
$2i$
$1i$
$-1i$
$-2i$
$-3i$
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7

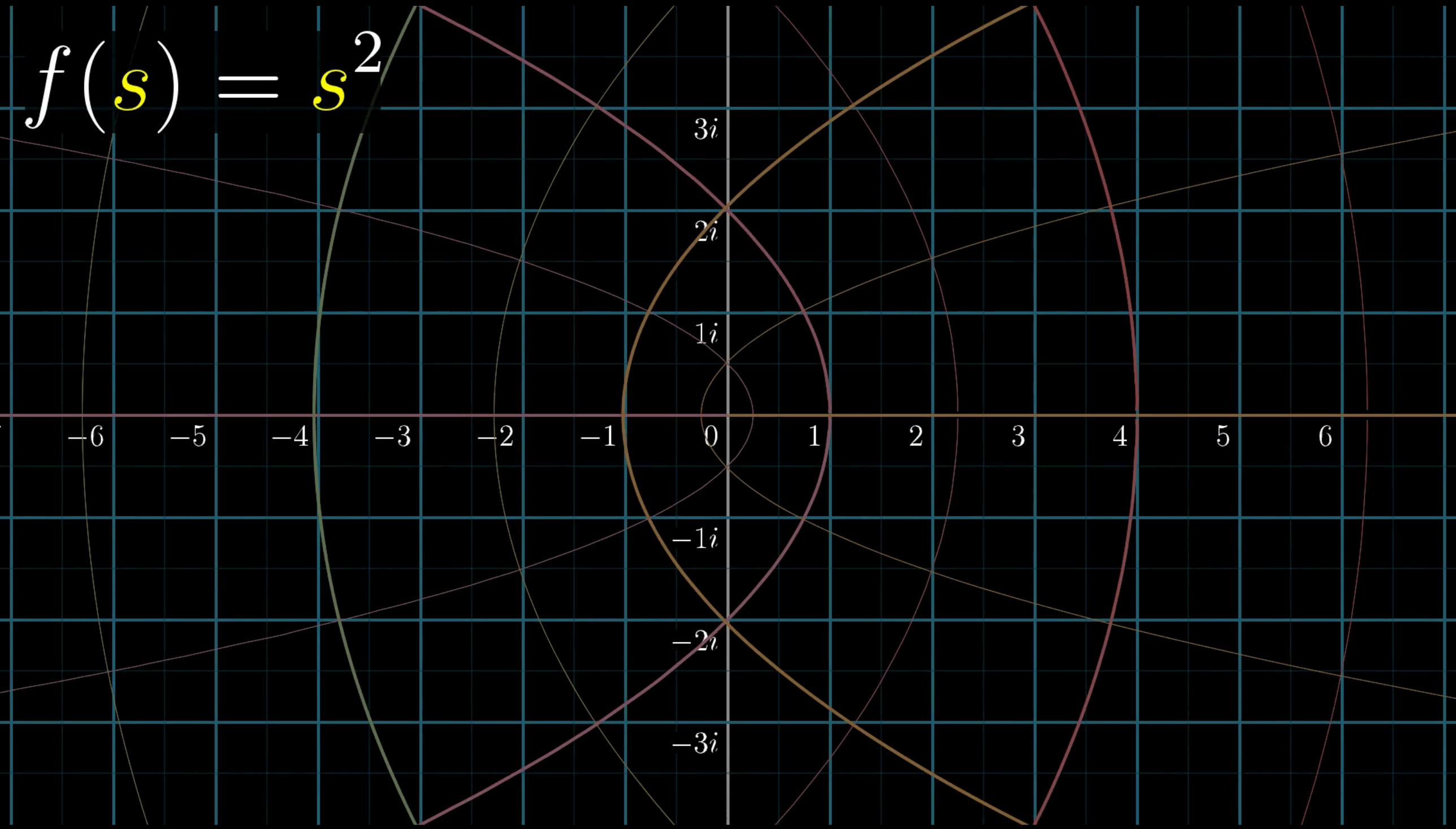

3i
2i
1i
−1i
−2i
−3i
−6
−5
−4
−3
−2
−1
2
3
4
5
6
7


3i
2i
1i
−1i
−2i
−3i
−6
−5
−4
−3
−2
−1
2
3
4
5
6
7


$3i$
$2i$
$1i$
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
$-1i$
$-2i$
$-3i$

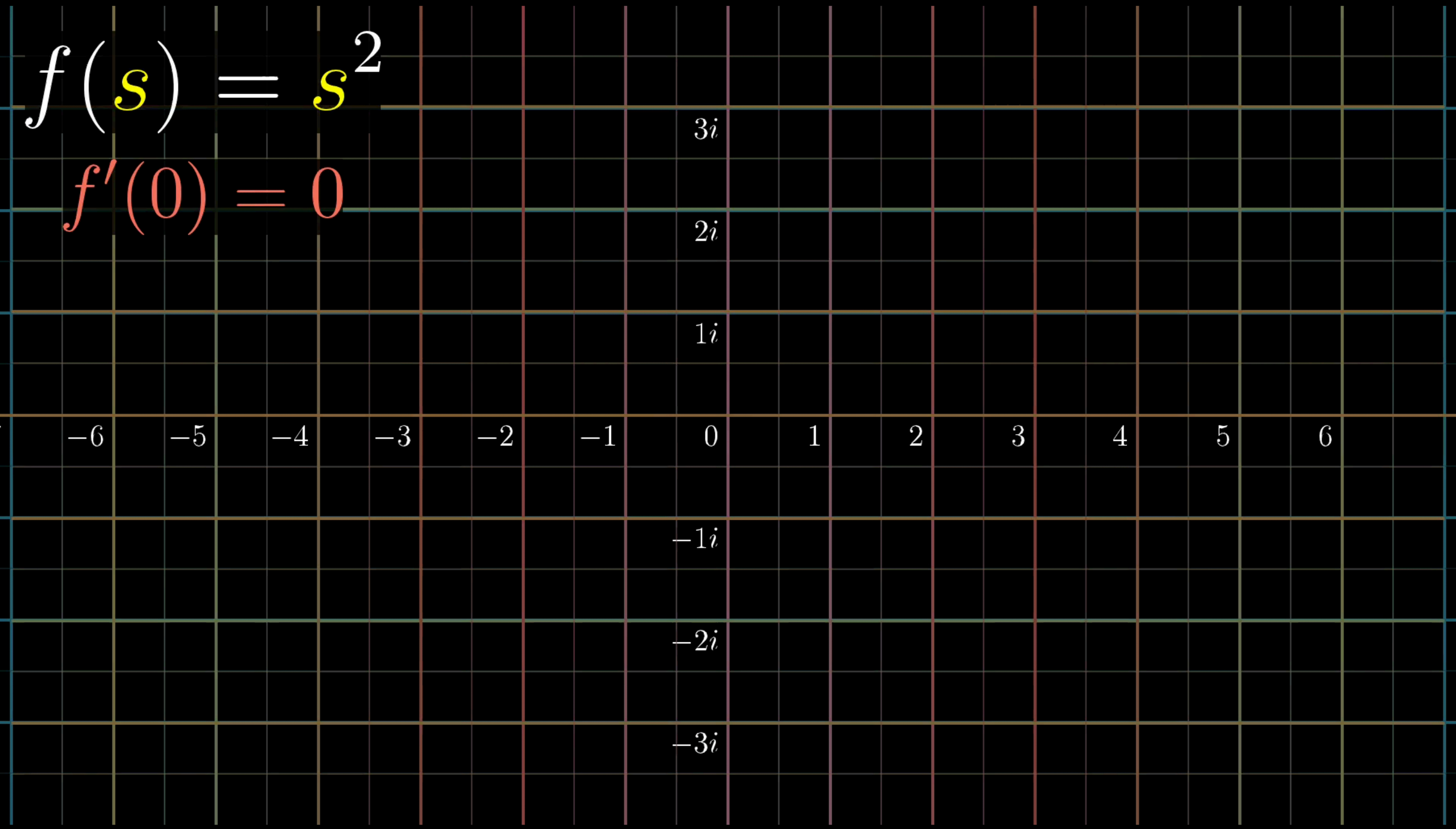
$$f(s) = s^2$$



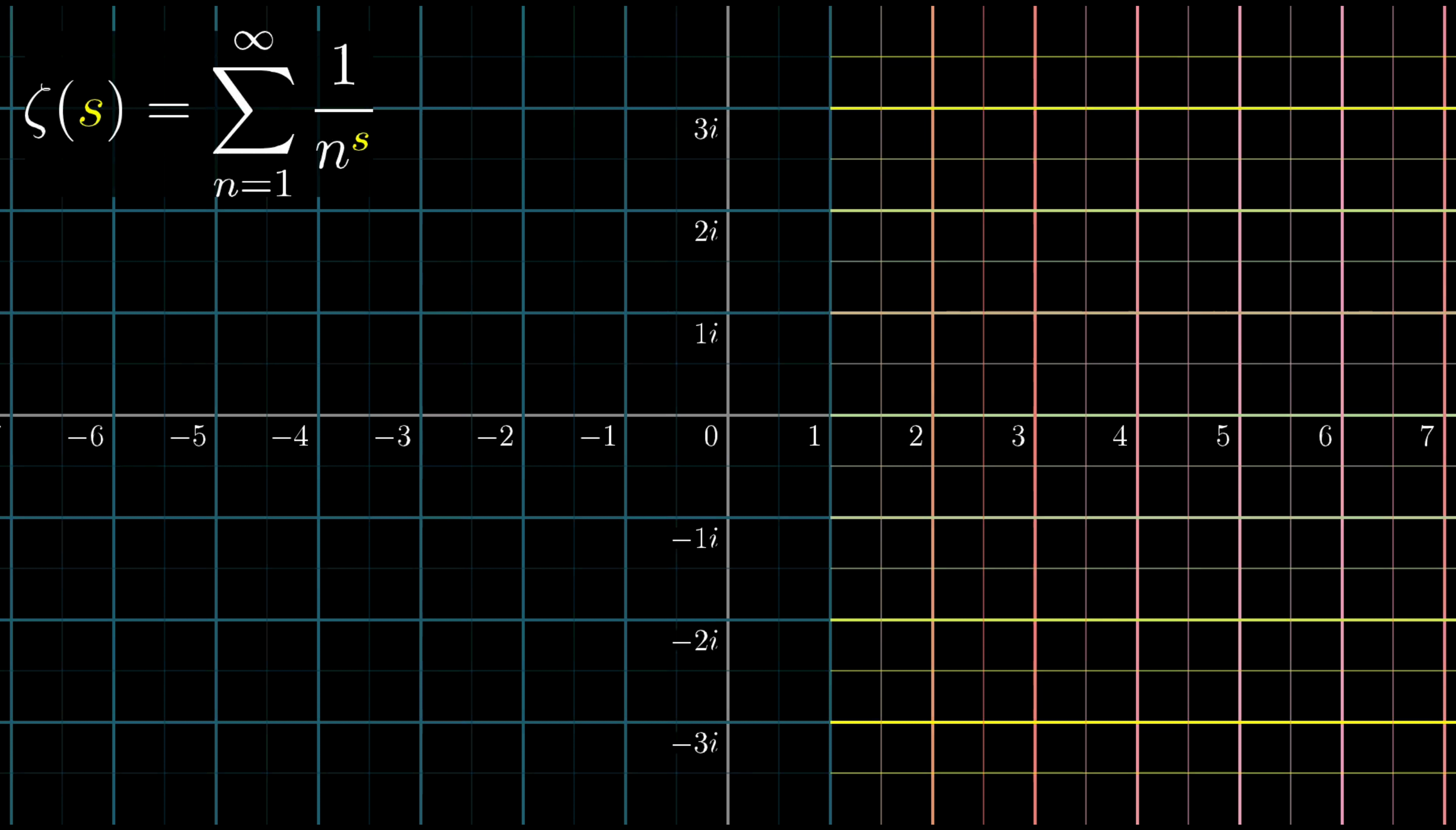

$f(s) = s^2$
$3i$
$2i$
$1i$
$-1i$
$-2i$
$-3i$
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6

$$f(s) = s^2$$

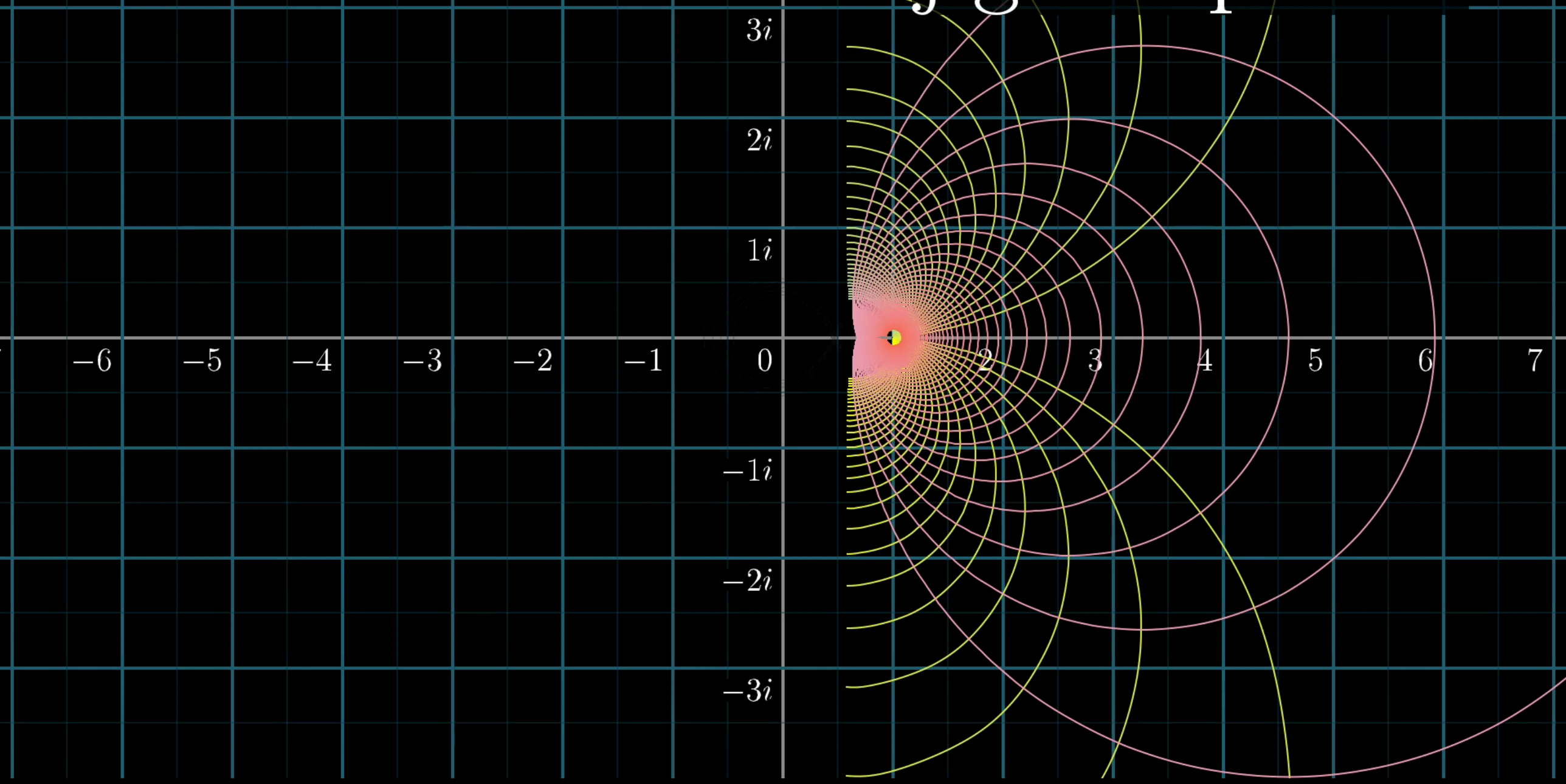
$$f'(0) = 0$$

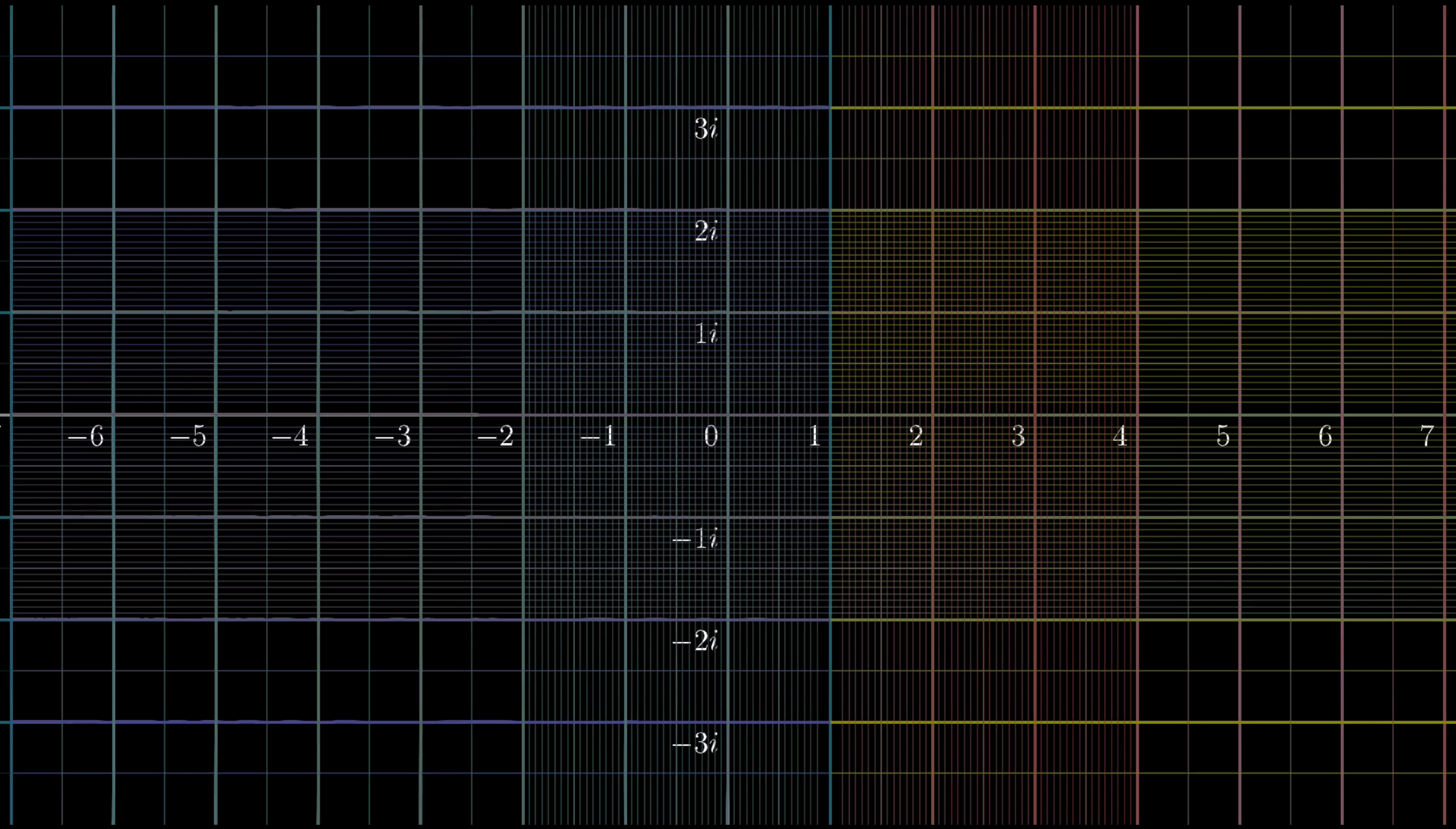


$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

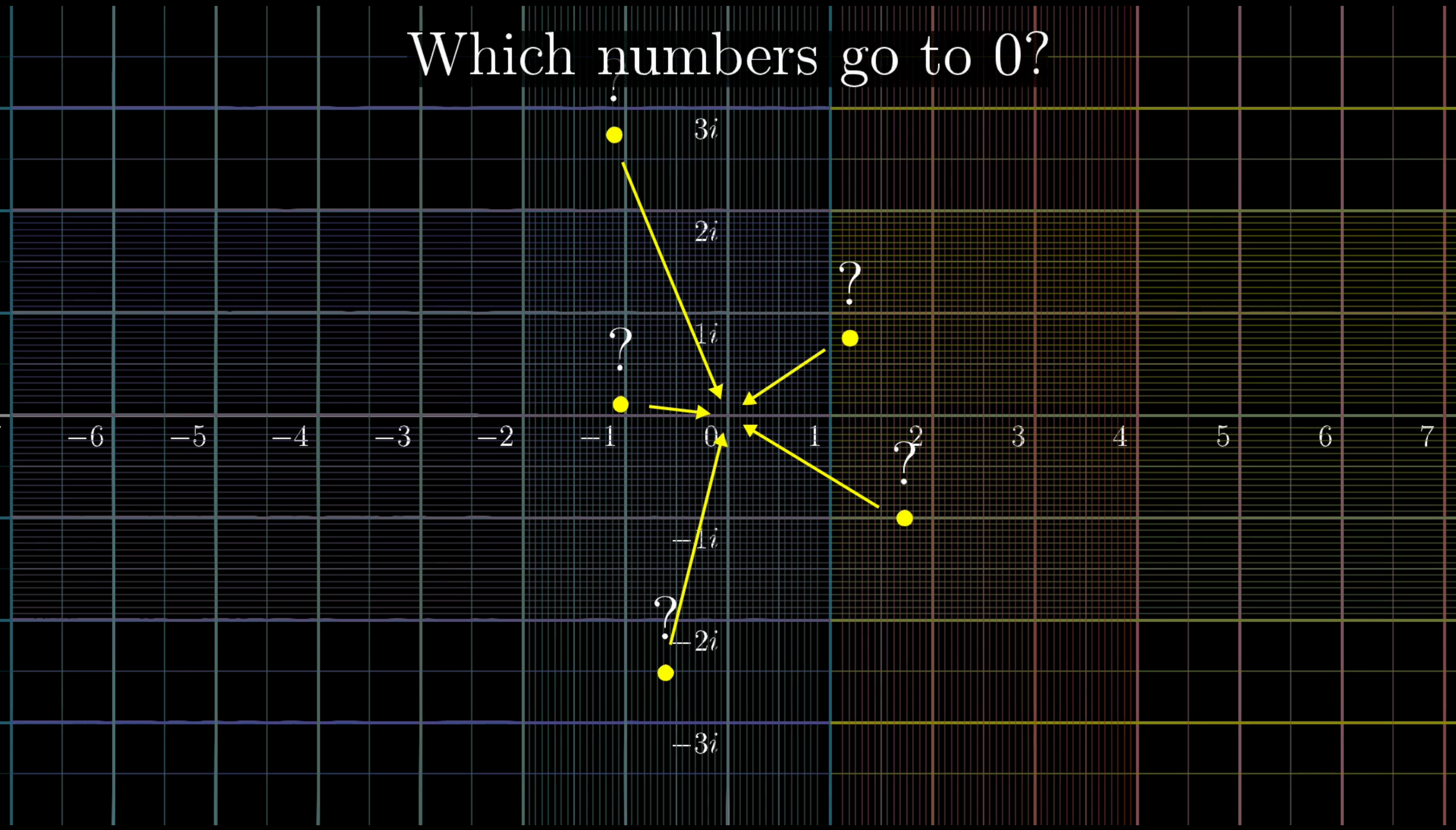


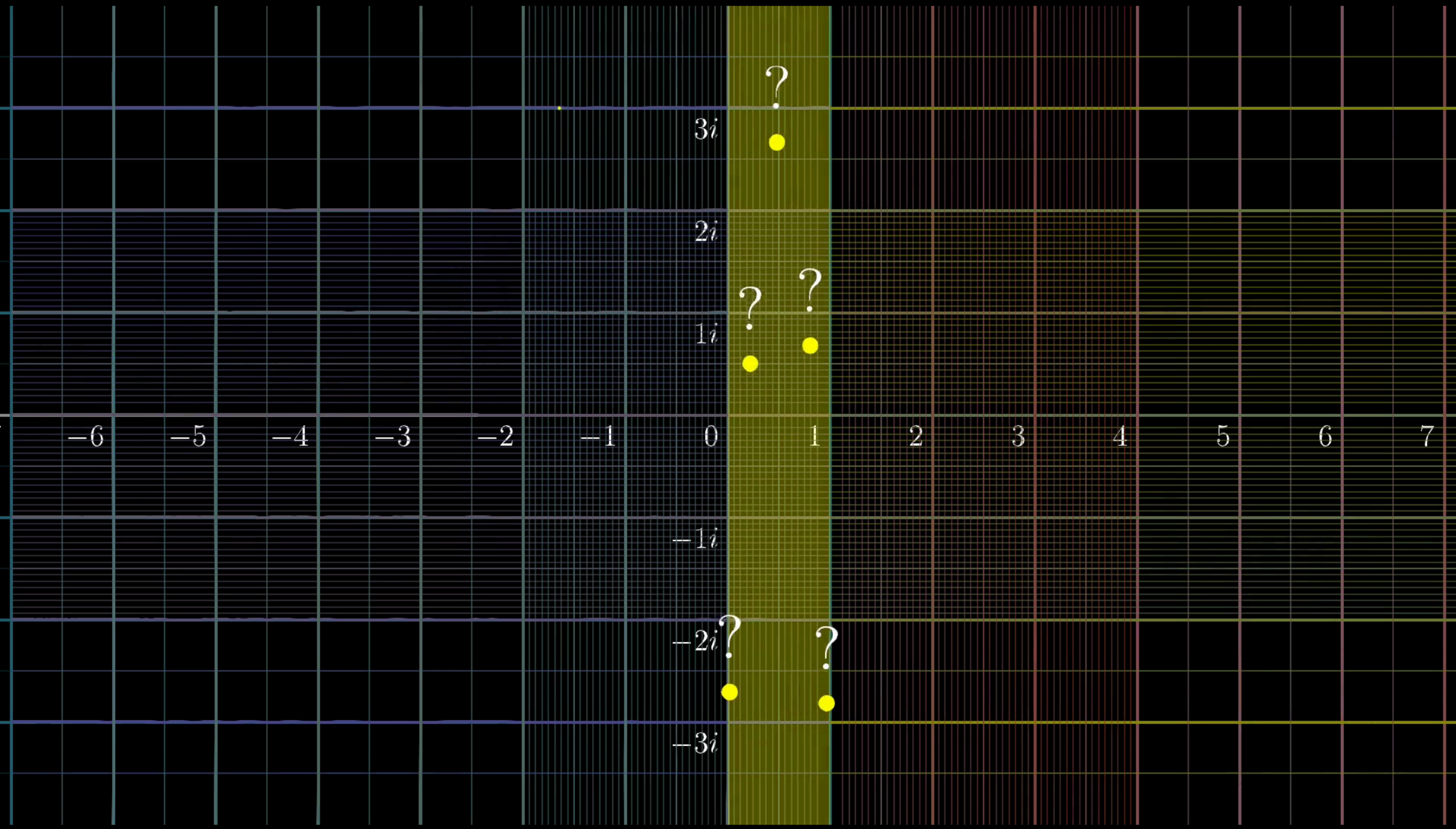
Infinite continuous jigsaw puzzle

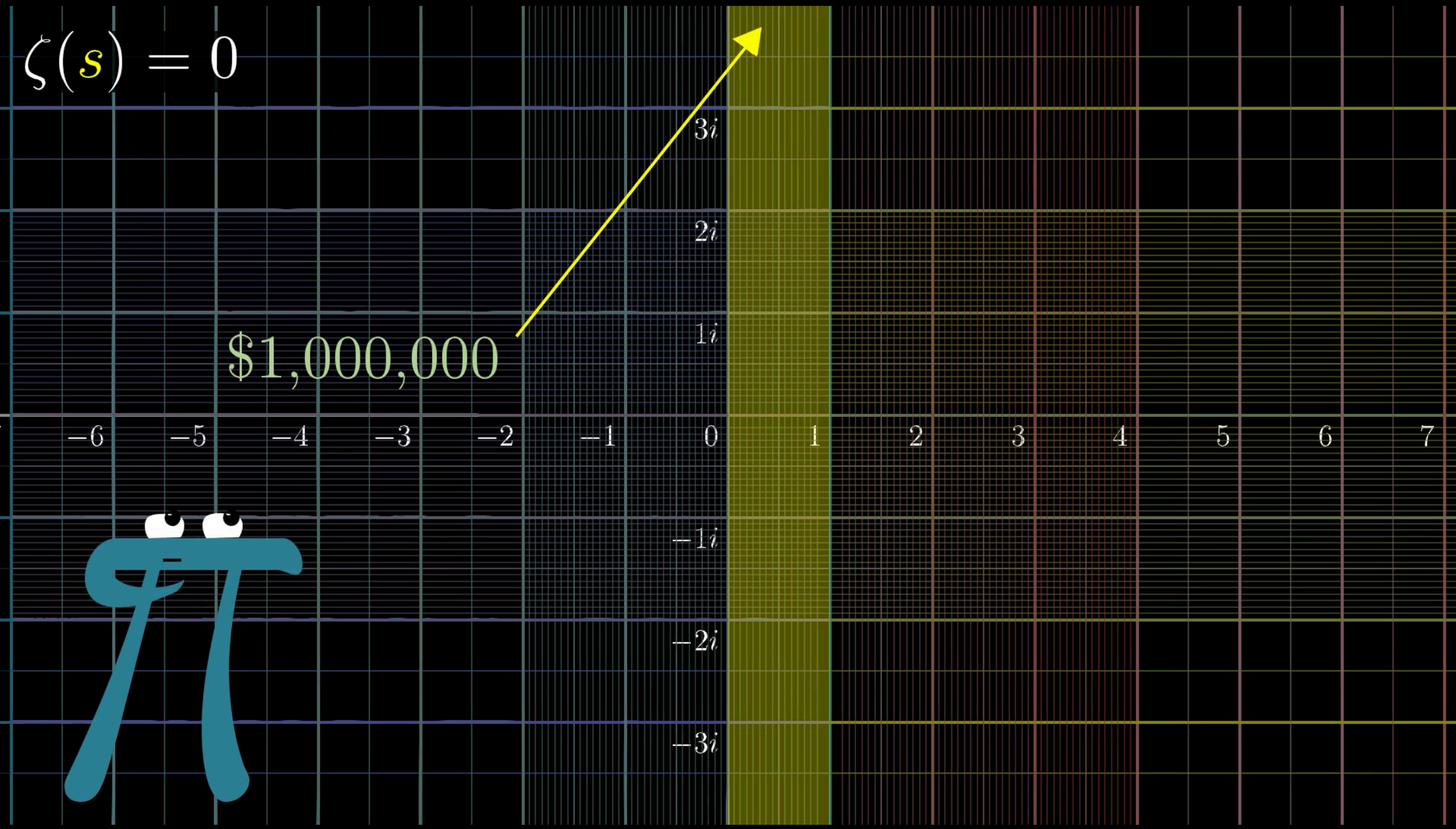


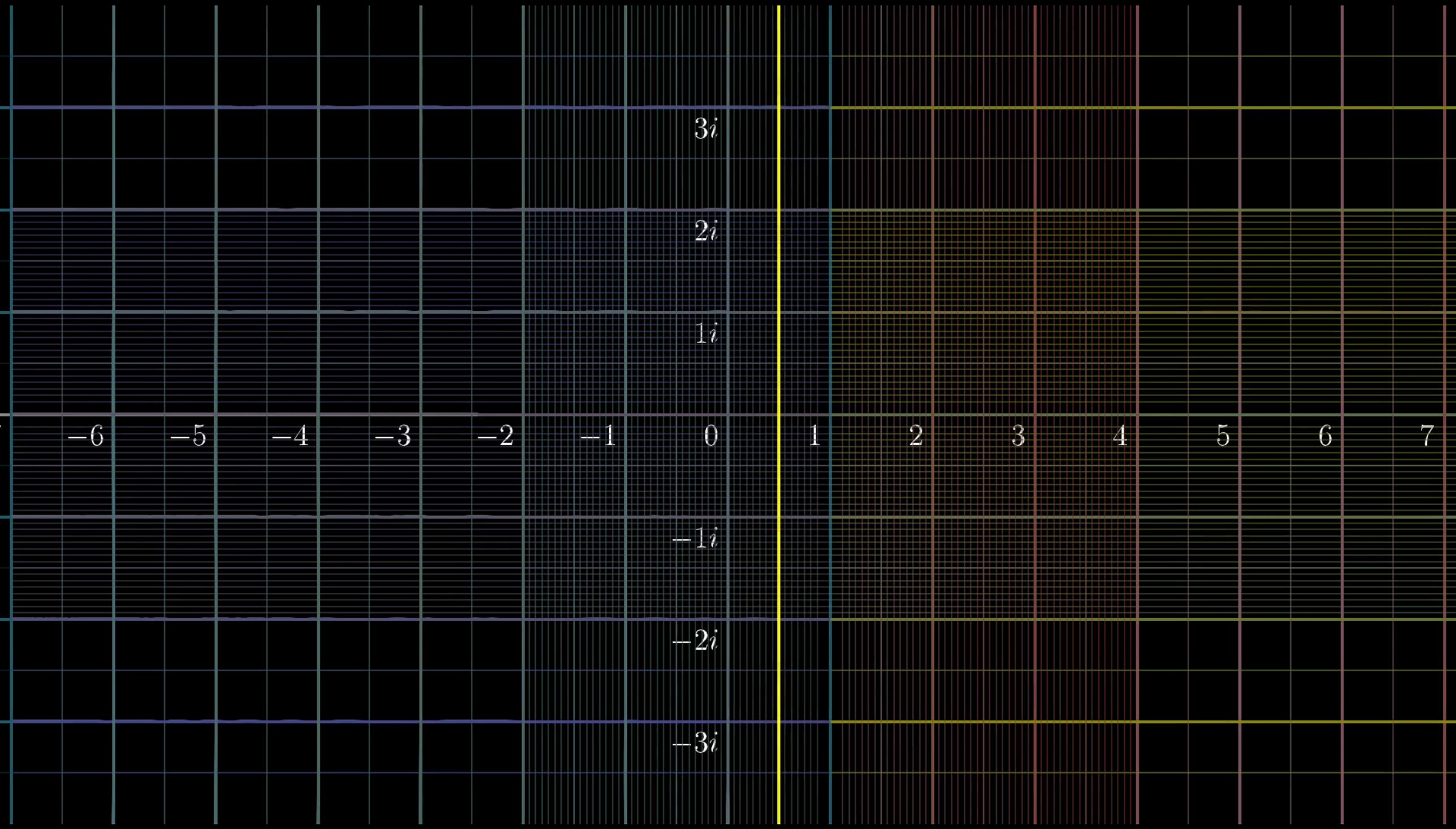

3i
2i
1i
−1i
−2i
−3i
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7

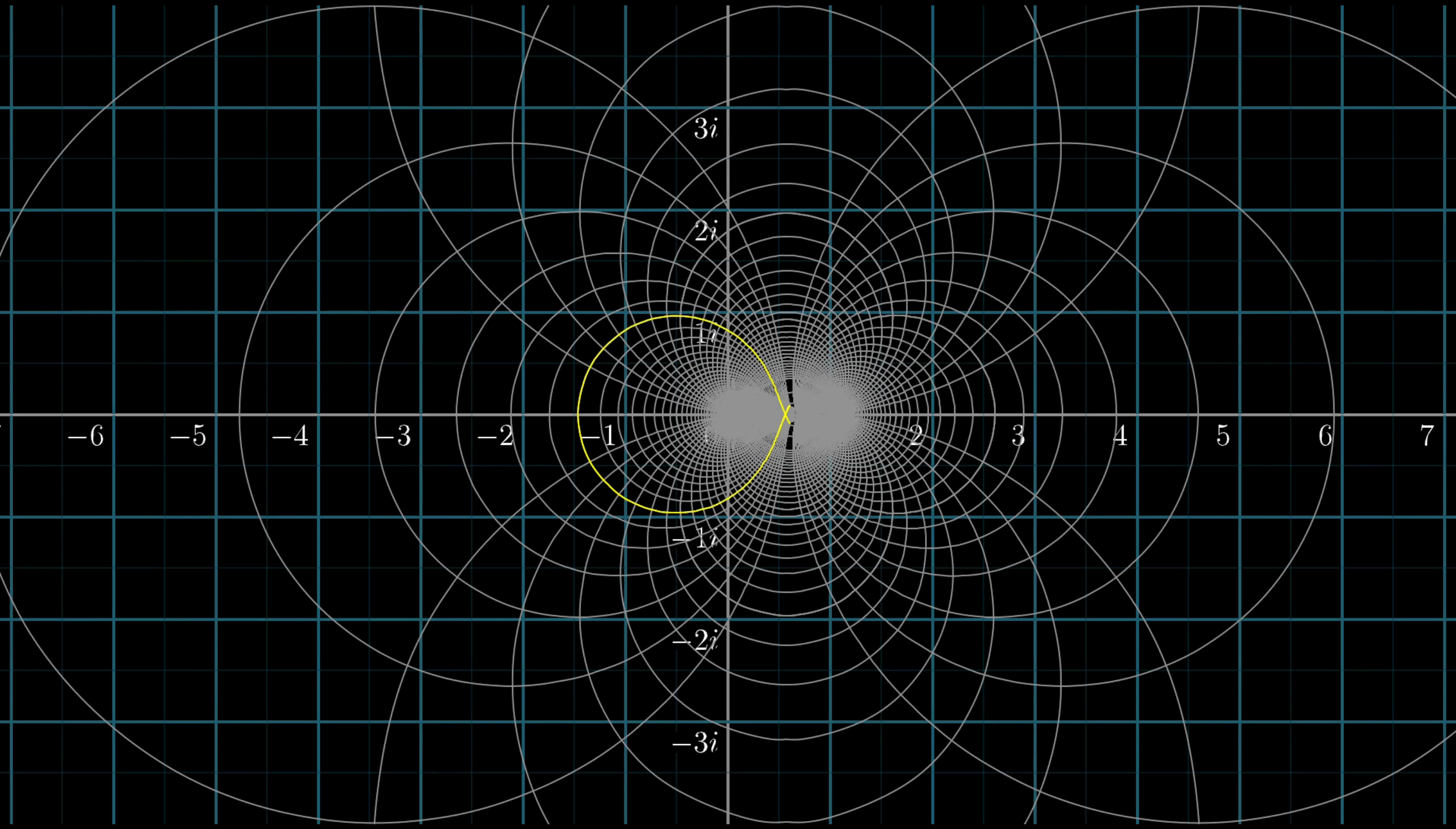
Which numbers go to 0?

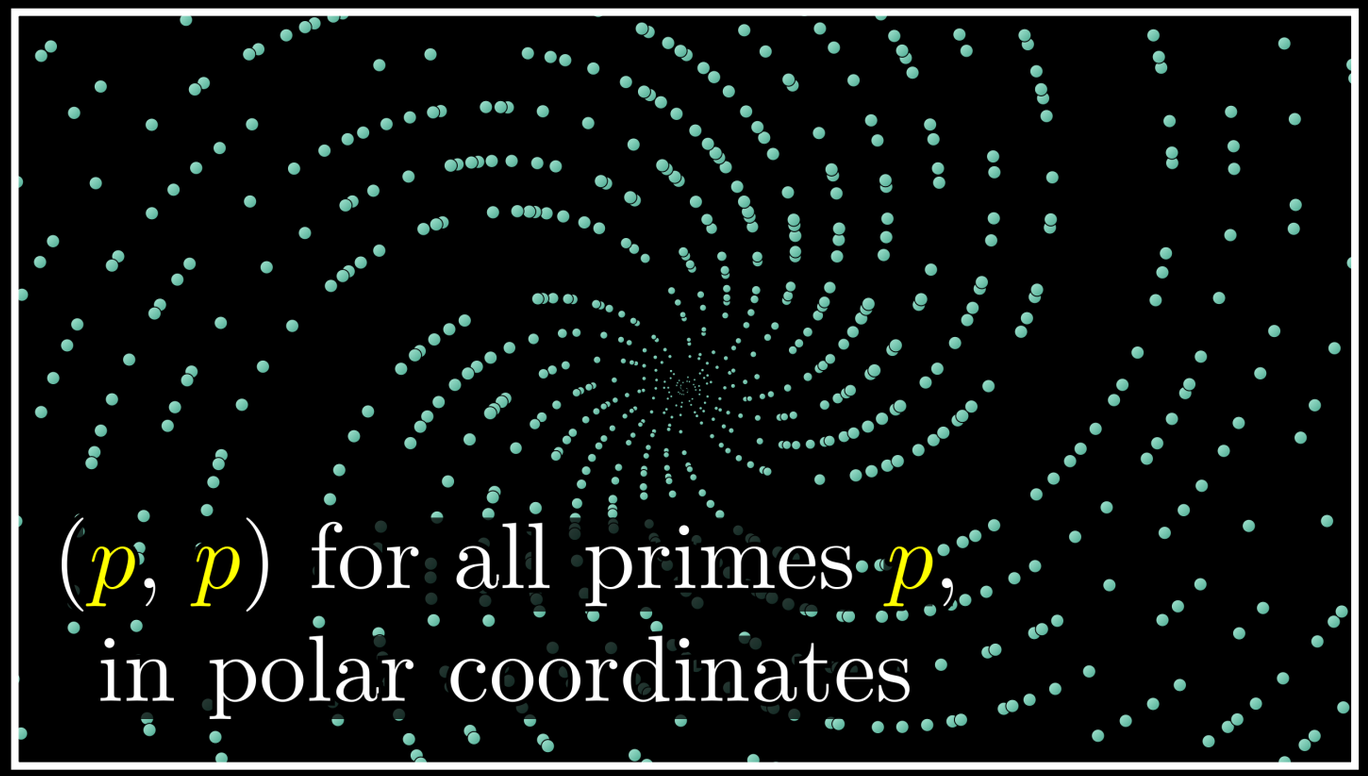



?
3i
2i
?
?
1i
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7
−1i
−2i
?
?
−3i

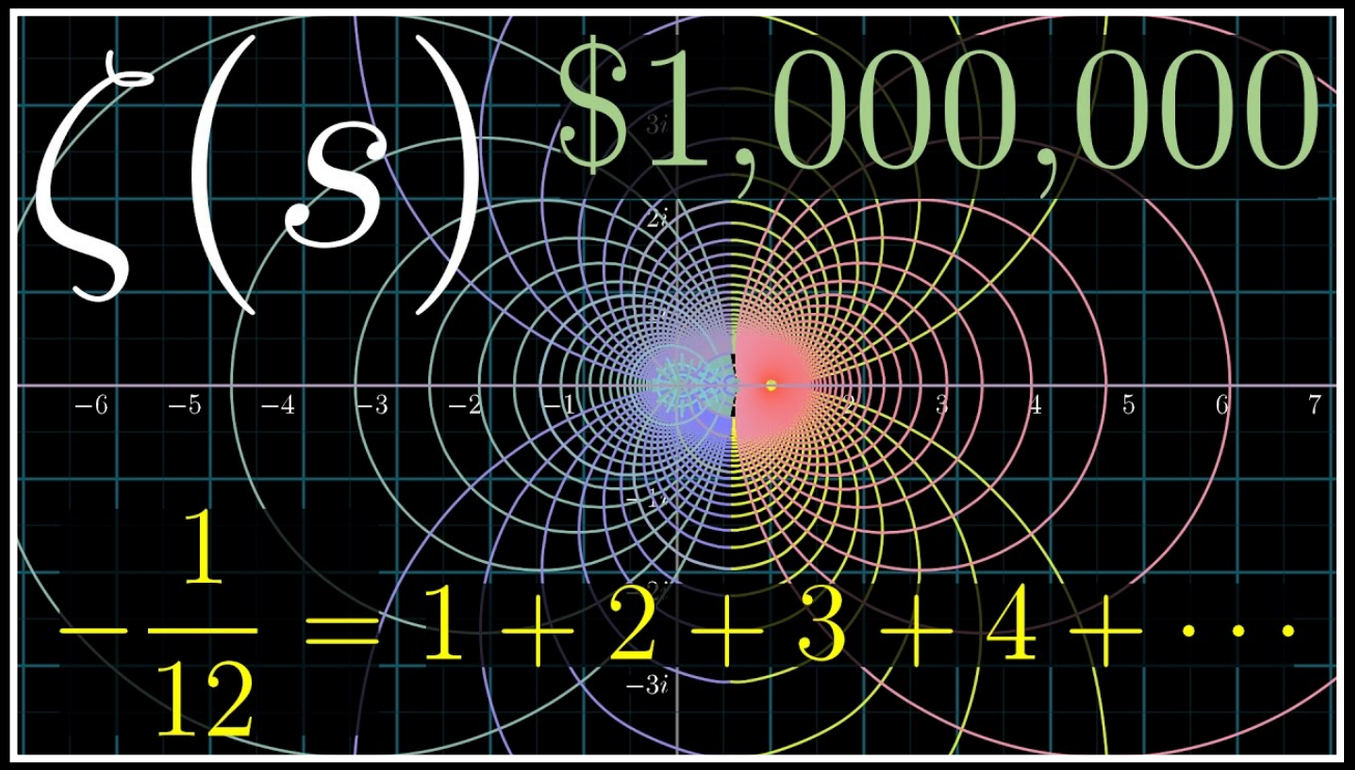

$\zeta(s) = 0$
$3i$
$2i$
$1i$
$\$1{,}000{,}000$
-6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7
$-1i$
$-2i$
$-3i$


3i
2i
1i
−1i
−2i
−3i
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
7


3i
2i
1i
−1i
−2i
−3i
−6
−5
−4
−3
−2
−1
2
3
4
5
6
7



Use visuals to inspire



Use visuals to elucidate

Thank you